Worst Case Pricing of
Rainbow Options

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Abstract

Options on two underlyings are a common exotic product in the equity and FX derivatives market. The value of these kinds of options depends on the correlation of the two underlyings. We will present a model to compute a lower bound for the price of this option. The model, represented by a non-linear parabolic PDE, is implemented with finite elements in order to demonstrate the results with several derivatives from the European market.
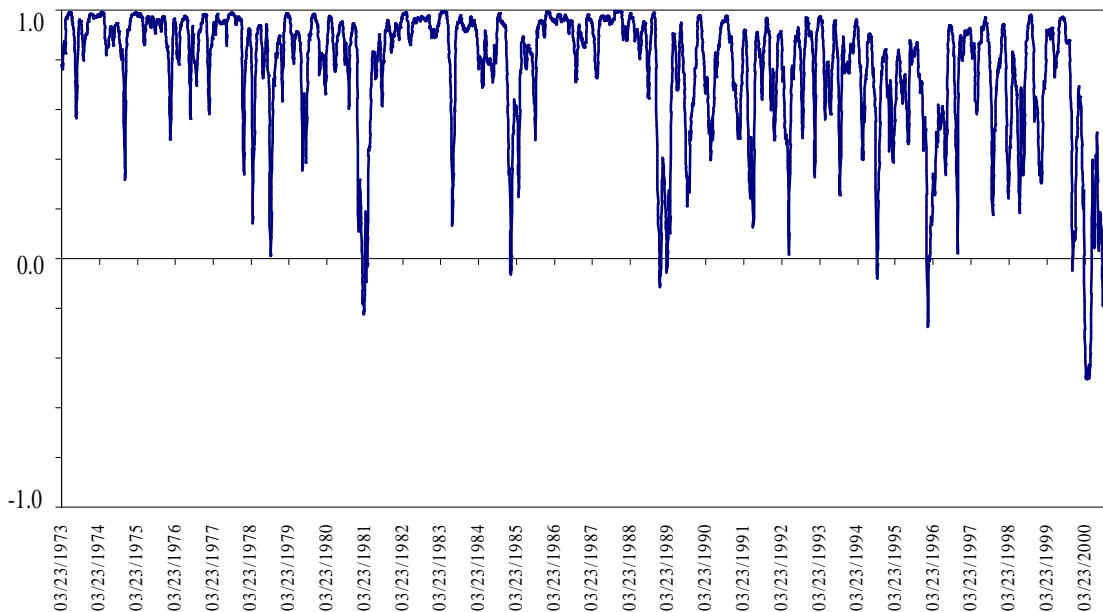
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Figure 1: Historical 60-day Correlation between Commerzbank and Dresdner Bank

1 Introduction

Rainbow options on two assets or currencies are common in the derivatives market. In the OTC market plain vanilla basket options and digital basket options, sometimes with barriers or caps, are common exotics. Options on-the-worst-of-two are sometimes embedded in triple currency linked bonds or reverse convertibles. Triple linked currency bonds can be divided into a straight bond and a series of puts on the worst of two currencies [12]. So-called double reverse convertibles can be seen as a straight bond (sometimes a zero coupon bond) and a short position in a put on the worst of two equities. Products of this kind are currently popular in the retail market and are offered by *BHF Bank* and *HSBC Trinkaus und Burkhardt*. Reverse convertibles including a basket option on two equities as offered by *ABN Amro* have not gained any popularity.

The price of an option on two assets depends strongly on the correlation of these two assets denoted by ρ. A straightforward extension of the classical Black-Scholes framework for options on two assets leads to the following well–known PDE [7]

$$
\begin{aligned}
\frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \quad &+ \quad \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \\
(r - D_1)S_1 \frac{\partial V}{\partial S_1} + (r - D_2)S_2 \frac{\partial V}{\partial S_2} \quad &= \quad rV - \frac{\partial V}{\partial t}
\end{aligned}
\tag{1}
$$

The problem, however, is that correlation ρ is not a constant as can be seen in fig. 1. Instead of giving a correct figure for the correlation, it is much easier to derive a reasonable guess for the lower and upper bound. In the FX market, it is possible to compute *implied* correlation [9]. This, however, is usually not possible in the equity market.

In the following, we will show how to price an option when only upper and lower bounds for correlation are known. This approach uses similar hedging arguments as the original approach by Black and Scholes [1]. We are only interested in the worst-case solution. The derived pricing PDE is nonlinear in contrast to eq. (1). This model is similar in spirit to earlier models with respect to other unknown parameters such as volatility, dividends, and interest rates. For a survey see [19]. We will show how to implement this model using a collocation finite element method. This includes a thorough discussion of the appropriate boundary conditions. As examples, we present numerical results for several rainbow options from the European market.

2   A Pricing Model with Uncertain Correlation

2.1   A Worst-Case Pricing Equation

Assuming that a lower and an upper bound for correlation is

$$-1 \le \rho^- \le \rho \le \rho^+ \le 1 \tag{2}$$

we shall use a similar argument as Black and Scholes [1] to derive a model for worst case pricing. Both assets follow the usual SDE:

$$dS_1 \;=\; \mu_1\, S_1\, dt \;+\; \sigma_1\, S_1\, dX \tag{3}$$
$$dS_2 \;=\; \mu_2\, S_2\, dt \;+\; \sigma_2\, S_2\, dX \tag{4}$$

Both assets are correlated

$$E\left(dX_i dX_j\right) = \rho\, dt \tag{5}$$

Setting up a portfolio consisting of a long position in one option and short positions in both underlyings leads to:

$$\Pi = V(S_1, S_2, t) - \Delta_1 S_1 - \Delta_2 S_2 \tag{6}$$

With Itô's Lemma an infinitesimal change in this portfolio can be expressed as:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right) dt + \sum_{i=1}^{2}\left(\frac{\partial V}{\partial S_i} - \Delta_i \right) dS_i \tag{7}$$

This expression can be simplified because $i, j = 2$. We also know that $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}$. To simplify the notation we set $\rho_{12} = \rho_{21} = \rho$.

$$\begin{aligned}
d\Pi \;=\;& \left(\frac{\partial V}{\partial t} + \sigma_1 \sigma_2 \rho S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right) dt \\
&+ \left(\frac{\partial V}{\partial S_1} - \Delta_1 \right) dS_1 + \left(\frac{\partial V}{\partial S_2} - \Delta_2 \right) dS_2
\end{aligned} \tag{8}$$

Choosing $\Delta_i = \frac{\partial V}{\partial S_i}$ for $i = 1, 2$ elimininates the risk just as in the classical argument when deriving the multidimensional Black-Scholes equation.

$$d\Pi = \left(\frac{\partial V}{\partial t} + \sigma_1 \sigma_2 \rho S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right) dt \tag{9}$$

In the Black-Scholes model it can be argued that from knowing V we also know $d\Pi$. Here, however, this is not true since ρ is unknown. Since we want to derive a worst-case-scenario model, we want to be extremely pessimistic: In every infinitesimal time step we assume a correlation that leads to the smallest growth in the portfolio. This implies that the return on this worst-case portfolio is set equal to the risk-free rate.

$$\min d\Pi \stackrel{!}{=} r\Pi\, dt \tag{10}$$

The minimization is with respect to ρ. Using eq. (6) the right-hand side of eq. (10) can be rewritten as

$$r\Pi\, dt = r\left(V(S_1, S_2, t) - \frac{\partial V}{\partial S_1}S_1 - \frac{\partial V}{\partial S_2}S_2 \right)\, dt \tag{11}$$

The left-hand side can be expressed as

$$\min_{\rho}\left[\left(\frac{\partial V}{\partial t} + \underbrace{\sigma_1 \sigma_2 \rho S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}}_{A} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right) dt \right] \tag{12}$$

Because of the minimization with respect to $\rho \in [\rho^-; \rho^+]$, we can distinguish two cases:

1. $\frac{\partial V}{\partial S_1 \partial S_2} = \Gamma_{cross} > 0 \quad \Rightarrow \quad A = \rho^- \sigma_1 \sigma_2 S_1 S_2 \Gamma_{cross}$

2. $\frac{\partial V}{\partial S_1 \partial S_2} = \Gamma_{cross} < 0 \quad \Rightarrow \quad A = \rho^+ \sigma_1 \sigma_2 S_1 S_2 \Gamma_{cross}$

Combining these results and taking dividends into account, we arrive at the worst-case pricing equation:

$$
\frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \quad + \quad \rho(\Gamma_{cross})\sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} \ +
$$

$$
(r - D_1)S_1 \frac{\partial V}{\partial S_1} + (r - D_2)S_2 \frac{\partial V}{\partial S_2} \quad = \quad rV - \frac{\partial V}{\partial t} \tag{13}
$$

$$
\rho(\Gamma_{cross}) \quad = \quad \begin{cases} \rho^- & \text{if } \Gamma_{cross} > 0 \\ \rho^+ & \text{if } \Gamma_{cross} < 0 \end{cases} \tag{14}
$$

$$
\Gamma_{cross} \quad = \quad \frac{\partial^2 V}{\partial S_1 \partial S_2} \tag{15}
$$

These results can be extended to ρ^- and ρ^+ as they are deterministic functions of t. Obviously, it has to hold that $\rho^-(t) < \rho^+(t)$. It is also possible to combine this model with other models to adjust for the uncertainty of the other unknown parameters.

2.2   Misspecification

This section deals with the error incurred by estimating the bandwidth of ρ in a too optimistic manner. We are interested in the effect of decreasing ρ^- and increasing ρ^+ by infinitesimal amounts. The change in V by relaxing ρ^- by ϵ is generally not the same as the change incurred by increasing ρ^+ to $\rho^+ + \epsilon$. Let us define:

$$
\delta_{\rho^-} := \frac{\partial V}{\partial \rho^-} \quad \approx \quad \frac{V(\rho^- - \epsilon) - V(\rho^- + \epsilon)}{2\epsilon} \tag{16}
$$

$$
\delta_{\rho^+} := \frac{\partial V}{\partial \rho^+} \quad \approx \quad \frac{V(\rho^+ - \epsilon) - V(\rho^+ + \epsilon)}{2\epsilon} \tag{17}
$$

$$
\text{with} \quad \epsilon \quad = \quad 0.01
$$

Both δ_{ρ^-} and δ_{ρ^+} are negative by definition: The price of an option in a worst-case scenario can only decrease by relaxing the input parameter.

3   Implementing the Model

3.1   A Numerical Approach: The Collocation Finite Element Method

These models are non-linear parabolic PDEs for which analytical solutions are not known. When, however, it can be shown that the Black-Scholes Γ_{cross} is single-signed, the non-linear PDE reduces to the Black-Scholes pricing equation which is linear. For certain rainbow options it is well-known that Γ_{cross} is single-signed so that only a linear PDE has to be solved. Sometimes, this can even be achieved analytically. From the large family of possible numerical solution techniques we have chosen a collocation finite element approach with cubic Hermite trial functions. This approximates Γ_{cross} accurately and delivers the other hedging parameters as a by-product. From the finite difference family only explicit FD are straightforward to employ; ADI and hopscotch methods need to be extended ([19], ch. 48; [15]). For a possible extension of FDs to compute accurate cross-derivatives see [18].

First we consider the stationary problem $L(u(x,y)) = f$, L being a non-linear differential operator. We look for an approximate solution $\tilde{u}(x,y)$ for the following problem:

$$
L(u) \quad = \quad f \ , \tag{18}
$$

$$
u(x_{min}) \quad = \quad u_{min_x}, \ \ u(x_{max}) = u_{max_x} \ , \tag{19}
$$

$$
u(y_{min}) \quad = \quad u_{min_y}, \ \ u(y_{max}) = u_{max_y} \ , \tag{20}
$$

That is, we consider a rectangular domain with Dirichlet conditions. This method can easily be generalized for non-rectangular domains and Neumann and mixed boundary conditions [14]. An approximate solution $\tilde{u}$ can take the following form:

$$\tilde{u} = \sum_{k=1}^{N} a_k \phi_k(x, y) \ . \tag{21}$$

The task is to find values for a_i which make $\tilde{u}$ the "best" approximation. While the Galerkin finite element methods sometimes used in two-asset derivative pricing (cf. [3], [16]) determine the a_i's by solving

$$\int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} \left(L(\tilde{u}) - f \right) \phi_k \ dy \ dx = 0, \ k = 1, \dots, N \ , \tag{22}$$

collocation finite element methods take the following approach. They enforce that, at certain points in the domain, the so-called *collocation points*, the PDE is exactly satisfied.

$$\left(L(\tilde{u}) - f \right) (\xi_k) = 0, \ k = 1, \dots, N \tag{23}$$

The domain is divided into rectangular disjoint elements, the *finite elements*. They are not necessarily of the same size. The four collocation points for each element $[x_i; x_{i+1}] \times [y_i; y_{i+1}]$ in the method used here are

$$\left\{ x_i + \left(\frac{1}{2} \pm \frac{1}{2\sqrt{3}} \right) (x_{i+1} - x_i); y_i + \left(\frac{1}{2} \pm \frac{1}{2\sqrt{3}} \right) (y_{i+1} - y_i) \right\} \tag{24}$$

Let s_i denote the number of gridlines in direction i. The approximate solution is defined by a linear combination of $4s_x s_y$ basis functions

$$H_i(x)H_j(y), \ H_i(x)S_j(y), \ S_i(x)H_j(y), \ S_i(x)S_j(y) \tag{25}$$

with

$$
\begin{aligned}
H_k(x) &= 3\left(\frac{x - x_{k-1}}{x_k - x_{k-1}} \right)^2 - 2\left(\frac{x - x_{k-1}}{x_k - x_{k-1}} \right)^3 \quad \text{for } x_{k-1} \leq x \leq x_k \\
&= 3\left(\frac{x_{k+1} - x}{x_{k+1} - x_k} \right)^2 - 2\left(\frac{x_{k+1} - x}{x_{k+1} - x_k} \right)^3 \quad \text{for } x_k \leq x \leq x_{k+1} \\
&= 0 \quad \text{elsewhere} \tag{26}
\end{aligned}
$$

$$
\begin{aligned}
S_k(x) &= -\frac{(x - x_{k-1})^2}{(x_k - x_{k-1})} + \frac{(x - x_{k-1})^3}{(x_k - x_{k-1})^2} \quad \text{for } x_{k-1} \leq x \leq x_k \\
&= \frac{(x_{k+1} - x)^2}{(x_{k+1} - x_k)} - \frac{(x_{k+1} - x)^3}{(x_{k+1} - x_k)^2} \quad \text{for } x_k \leq x \leq x_{k+1} \\
&= 0 \quad \text{elsewhere} \tag{27}
\end{aligned}
$$

$$
\begin{aligned}
H_k(y) &= 3\left(\frac{y - y_{k-1}}{y_k - y_{k-1}} \right)^2 - 2\left(\frac{y - y_{k-1}}{y_k - y_{k-1}} \right)^3 \quad \text{for } y_{k-1} \leq y \leq y_k \\
&= 3\left(\frac{y_{k+1} - y}{y_{k+1} - y_k} \right)^2 - 2\left(\frac{y_{k+1} - y}{y_{k+1} - y_k} \right)^3 \quad \text{for } y_k \leq y \leq y_{k+1} \\
&= 0 \quad \text{elsewhere} \tag{28}
\end{aligned}
$$

$$
\begin{aligned}
S_k(y) &= -\frac{(y - y_{k-1})^2}{(y_k - y_{k-1})} + \frac{(y - y_{k-1})^3}{(y_k - y_{k-1})^2} \quad \text{for } y_{k-1} \leq y \leq y_k \\
&= \frac{(y_{k+1} - y)^2}{(y_{k+1} - y_k)} - \frac{(y_{k+1} - y)^3}{(y_{k+1} - y_k)^2} \quad \text{for } y_k \leq y \leq y_{k+1} \\
&= 0 \quad \text{elsewhere} \tag{29}
\end{aligned}
$$

The approximate solution $\tilde{u}(x, y)$ has the form

$$\tilde{u}(x, y) = \sum_{i=1}^{s_x} \sum_{j=1}^{s_y} \left\{ A_{ij}H_i(x)H_j(y) + B_{ij}H_i(x)S_j(y) + C_{ij}S_i(x)H_j(y) + D_{ij}S_i(x)S_j(y) \right\} \tag{30}$$

so the $N = 4s_x s_y$ unknowns are A_{11}, B_{11}, C_{11}, D_{11}, A_{12}, B_{12}, C_{12}, D_{12}, ..., $A_{s_x s_y}$, $B_{s_x s_y}$, $C_{s_x s_y}$, $D_{s_x s_y}$ which are relabeled to $a_1, \ldots, a_N$ for notational convenience. The approximating function $\tilde{u}(x,y)$ has a continuous mixed derivative $\tilde{u}_{xy}$ because the mixed derivative of each of the basis functions is continuous. To be succesful in this setting, the key feature of continuous mixed derivatives has to hold. This feature is not common to most FE methods in use today. The basis functions normally used by Galerkin methods do not even have continuous $\tilde{u}_x$ or $\tilde{u}_y$.

The approximate solution is required to satisfy the PDE exactly at the four collocation points, in each of the $(s_x - 1)(s_y - 1)$ subrectangles, and to satisfy the boundary conditions at certain points. The number of boundary collocation points plus the number of interior collocations points $4(s_x - 1)(s_y - 1)$, is equal to the number of basis functions $4s_x s_y$, which is equal to the number of unknowns N. Finding the N parameters $a_1, \ldots, a_N$ results in a system of non-linear equations:

$$
\begin{aligned}
f_1(a_1, \ldots, a_N) &= 0 \\
&\vdots \\
f_N(a_1, \ldots, a_N) &= 0
\end{aligned}
\tag{31}
$$

This system of non-linear equations is solved via Newton's method. The integration of time is similar to the Galerkin finite element method. Spatial variables are discretized with finite elements while time is treated with finite differences. This can be visualized as the non-linear elliptic operator $L(u)$ evolving through time. Each equation corresponds to a collocation point. The dynamic counterpart to eq. (31) is given by

$$
\dot{a} = \left(\begin{array}{c} \dot{a}_1 \\ \vdots \\ \dot{a}_N \end{array} \right) = \left(\begin{array}{c} g_1(a_1, \ldots, a_N) \\ \vdots \\ g_N(a_1, \ldots, a_N) \end{array} \right)
\tag{32}
$$

This stiff system of non-linear *ordinary* differential equations can be solved with various time-stepping procedures. We have chosen a Crank-Nicolson implementation. The initial conditions to eq. (32) are given by a discretization of the final condition of the PDE. All computations have been performed with PDE2D, a general purpose finite element solver described in [14].

3.2 Estimating the Data

Normally, there are not enough options in the market to compute *implied* correlations from market data. When the option is exchange-traded, it is possible to construct a historical data series of implicit correlations. As a first approach, one can pick the extremes of this time series as ρ_{max} and ρ_{min}. For OTC products, a time series of historical correlations as in fig. 1 gives some insights into the behavior of correlation. Another approach to find ρ_{min} and ρ_{max} is based on the forward Kolmogorov equation [11]. First, one has to specify the functions $a(\rho)$ and $b(\rho)$ in

$$
d\rho_t = a(\rho_t)dt + b(\rho_t)dX_t
\tag{33}
$$

and estimate their parameters. The forward Kolmogorov equation with appropriate boundary and final conditions gives the probabilities that ρ will breach a pre-specified band. The converse approach is also valid: The correlation distribution is determined at each time step. In case one does not want the correlation to breach the band with a 5 % probability, one has to cut off the 0.025 quantiles at the left and right hand side of the distribution. These quantiles are used as lower and upper bounds. Usually, the lower and upper bounds are both functions of time.

4 Applications of the Model

4.1 Capped Calls on Baskets

As a first example of a rainbow option, we consider a capped call on a basket of two equities. This is a common OTC product. Sometimes it is traded at exchanges as well.[1] For pricing this call we have to solve PDE (13) - (15) with the following final and boundary conditions:

$$V(T) \quad = \quad \min\left[\text{cap}, \max\left(0, w_1 S_1 + w_2 S_2 - E\right)\right] \tag{34}$$

$$\Omega \quad = \quad [S_1^{min}; S_1^{max}] \times [S_2^{min}; S_2^{max}] \tag{35}$$

$$V(S_1, S_2^{min}) \quad = \quad BS\left(S_1, \frac{E}{w_1}, t\right) - BS\left(S_1, \text{cap}, t\right) \tag{36}$$

$$\frac{\partial V(S_1, S_2^{max})}{\partial S_2} \quad = \quad 0 \tag{37}$$

$$V(S_1^{min}, S_2) \quad = \quad BS\left(S_2, \frac{E}{w_1}, t\right) - BS\left(S_2, \text{cap}, t\right) \tag{38}$$

$$\frac{\partial V(S_1^{max}, S_2)}{\partial S_1} \quad = \quad 0 \tag{39}$$

The function $BS(x, y, t)$ denotes the price of a European call option with an underlying price of x and a strike price of y as given by the Black-Scholes model [1].[2]

Parameter	Symbol	Value
First asset price	S_1	100
Weight first asset	w_1	1
Second asset price	S_2	100
Weight second asset	w_2	1
Strike price	E	200
Interest rate	r	0.0953102
Dividend yield first asset	D_1	0.0487902
Dividend yield second asset	D_2	0.0
Time to Maturity	T	0.5
Domain	S_1^{min}	0
	S_1^{max}	200
	S_2^{min}	0
	S_2^{max}	200
Cap	cap	10

Table 1: Data Capped Call on a Basket

Volatility		Scenario 1	Scenario 2	Scenario 3	
σ_1	σ_2	$\rho = 0.5$	$\rho^- = 0.4 \leq \rho \leq 0.6 = \rho^+$	$\rho^- = 0.3 \leq \rho \leq 0.7 = \rho^+$	δ_{ρ^-}
	0.1	5.3676	5.1953	5.0833	-0.050
0.1	0.2	4.9923	4.8832	4.7742	-0.050
	0.3	4.7555	4.6612	4.5677	-0.045
	0.1	4.9625	4.8878	4.7767	-0.050
0.2	0.2	4.9989	4.7348	4.5998	-0.065
	0.3	4.7120	4.5884	4.4569	-0.065
	0.1	4.7632	4.6669	4.5706	-0.050
0.3	0.2	4.7246	4.5686	4.4584	-0.070
	0.3	4.6329	4.5006	4.3591	-0.070

Table 2: Results Capped Call on a Basket

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[1] In Germany, examples for capped options are WKN 822361, WKN 822362, WKN 822380, and WKN 822399 which are traded in Frankfurt, Düsseldorf, Stuttgart.

[2] For a discussion of the boundary conditions and alternative specifications see [16].

4.2   Options on the Maximum or the Minimum of Two Risky Assets

This product has the payoff

$$[\phi(\eta \min(\eta S_1, \eta S_2) - K)]^+ \tag{40}$$

This is a call ($\phi = +1$) or put ($\phi = -1$) on the maximum ($\eta = -1$) or minimum ($\eta = +1$) of the two underlyings S_1 and S_2 with a strike K. It has been shown by ([13], eq. (123)) that Γ_{cross} can be expressed as:

$$\Gamma_{cross} = \phi\eta \underbrace{\frac{e^{(r-D_1)(T-t)}}{S_2\sigma\sqrt{T-t}}}_{=B} n(d_3)\ N\left(\phi\sigma\frac{d_3\rho_1 - d_1}{\sigma_2\sqrt{1-\rho^2}}\right) \tag{41}$$

$$\rho_1 = \frac{\rho\sigma_2 - \sigma_1}{\sigma} \tag{42}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \tag{43}$$

$$d_1 = \frac{\ln\left(\frac{S_1}{E}\right) + \left(r - D_1 + \frac{1}{2}\sigma_1^2\right)(T-t)}{\sigma_1\sqrt{T-t}} \tag{44}$$

$$d_3 = \frac{\ln\left(\frac{S_2}{S_1}\right) + \left(D_1 - D_2 - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{45}$$

The functions $N(\cdot)$ and $n(\cdot)$ represent the cumulative distribution function and the density function of the standard normal distribution, respectively. Since B, $N(\cdot)$ and $n(\cdot)$ are positive, the sign of Γ_{cross} depends solely on the variables ϕ and η which determine the product type.

Product	ϕ	η	Γ_{cross}
Call on the Maximum	$+1$	-1	$'-'$
Call on the Minimum	$+1$	$+1$	$'+'$
Put on the Minimum	-1	$+1$	$'-'$
Put on the Maximum	-1	-1	$'+'$

Table 3: Signs of Γ_{cross}

Using these results, the worst-case scenarios can be computed using the Black-Scholes price with the following correlations [2]:

Product	Correlation
Call on the Maximum	$\rho^+_{Average} = \frac{1}{T-t}\int_t^T \rho^+(\tau)d\tau$
Call on the Minimum	$\rho^+_{Average} = \frac{1}{T-t}\int_t^T \rho^+(\tau)d\tau$
Put on the Minimum	$\rho^-_{Average} = \frac{1}{T-t}\int_t^T \rho^-(\tau)d\tau$
Put on the Maximum	$\rho^-_{Average} = \frac{1}{T-t}\int_t^T \rho^-(\tau)d\tau$

Table 4: Worst-Case Correlations

In the case when $\rho^+(t)$ is a constant, $\rho^+_{Average}$ reduces to ρ^+. This holds in a similar fashion for $\rho^-(t)$.

4.3   Two-Asset Barrier Options

In October 1993, Bankers Trust structured a call option on a basket of Belgian stocks that would knock out if the Belgian franc -not the Belgian stock- appreciated by more than 3.5 %. So we have two spatial variables entering the pricing equation: one that determines the payoff treating the basket as a single risk factor, and one that determines whether the option is still alive. This kind of product went through a boom in the OTC market in 1997 with a strike in the exchange rate of USD vs. DEM and a barrier in the exchange rate in USD vs. FRF. Back then, this product was employed to profit from the high difference in historical and implied correlation. Normally, this product comes with an FX rate on the knock-out barrier so that

an adjustment for discrete monitoring of the barrier is not necessary. This product can be generalized for knock-in and part-time barriers so that the barrier is only valid for parts of the life of the option [5].

We again solve the PDE eq. (13) - (15) with the following final and boundary conditions:

$$V(T) = \max(0, w_2 S_2 - E) \tag{46}$$

$$\Omega = [S_1^{min}; S_1^{max}] \times [S_2^{min}; S_2^{max}] \tag{47}$$

$$\frac{\partial V(S_1, S_2^{min})}{\partial S^2} = 0 \tag{48}$$

$$\frac{\partial V(S_1, S_2^{max})}{\partial S_2} = 1 \tag{49}$$

$$V(S_1^{min}, S_2) = 0 \tag{50}$$

$$\frac{\partial V(S_1^{max}, S_2)}{\partial S_1} = 0 \tag{51}$$

and the data:

Parameter	Symbol	Value
First asset price	S_1	100
Weight first asset	w_1	1
Weight second asset	w_2	1
Strike price	E	100
Interest rate	r	0.1
Dividend yield first asset	D_1	0.0
Dividend yield second asset	D_2	0.0
Volatility first asset	σ_1	0.2
Volatility second asset	σ_2	0.3
Time to Maturity	T	0.5
Correlation	ρ	varies
Domain	S_1^{min}	varies
	S_1^{max}	300
	S_2^{min}	0
	S_2^{max}	300

Table 5: Data Two-Asset Barrier Option Product

S_2	Location of Barrier					
	95			99		
	$\rho = 0.5$		$0.4 \leq \rho \leq 0.6$	$\rho = 0.5$		$0.4 \leq \rho \leq 0.6$
	Anal.	FE	FE	Anal.	FE	FE
80	1.3373	1.3353	1.2257	0.3570	0.3568	0.3143
85	2.1369	2.1320	1.9643	0.5572	0.5560	0.4945
90	3.1633	3.1566	2.9205	0.8080	0.8064	0.7233
95	4.4008	4.3939	4.0836	1.1039	1.1026	0.9969
100	5.8210	5.8155	5.4230	1.4376	1.4368	1.3090
105	7.3901	7.3874	6.9287	1.8007	1.8008	1.6524
110	9.0741	9.0753	8.5487	2.1859	2.1870	2.0199
115	10.8430	10.8485	10.2603	2.5868	2.5890	2.4051
120	12.6721	12.6821	12.9316	2.9984	3.0017	2.8030

Table 6: Results European Two-Asset Barrier Option

4.4 Basket Options

It holds for both calls and puts that the higher the correlation, the higher the price of the option. Consequently

$$\frac{\partial V}{\partial \rho} = \kappa \geq 0 \tag{52}$$

From the following relationship ([13], eq. (100))

$$\kappa = \underbrace{\sigma_1 \sigma_2 \tau S_1 S_2}_{\geq 0} \Gamma_{12} \tag{53}$$

we can conclude that Γ_{12} is always positive. Therefore, the worst-case price of a basket option only depends on ρ^-. Since no closed-form solution is known for basket options, the price of this type of option has to be computed numerically or by using an analytical approximation. For a survey of available methods see [6].

5   Conclusions and Outlook

For many rainbow options, the worst case scenarios cannot be computed by simply plugging in the worst case correlation into the Black-Scholes models. Whenever numerical techniques need to be used, they need to provide a good approximation of the mixed derivative. One method that does well under these circumstances is a collocation finite element method with a cubic Hermite basis function.

Other common OTC products are rainbows with a binary payoff structure and various caps or barriers. Digital payoffs lead to high curvature in the spatial variables, a feature finite elements can easily deal with. Barriers introduce non-rectangular domains (compare [16]) which are hard to cover with finite differences while they pose no problem for finite element methods. Recently, some American rainbows have been sold.[3] Currently, no models are available that describe the worst-case price in the presence of an early exercise. Also, since most basket options have more than just two underlyings, solvers in higher dimensions need to be developed.

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[3] For example: Deutsche Bank has issued various American options on baskets of Hewlett Packard Co./Agilent Technology (WKN 845950ff) and Roche Holding AG/Givaudan AG (WKN 836746ff) which are listed at various German stock exchanges.

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