Income Taxation, Government Expenditure, and Long–Run Stochastic Growth

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Diskussionspapier Nr. 220

März 1999

Abstract

This paper employs a stochastic endogenous growth model with productive government expenditure to analyze the macroeconomic effects of income taxation. We demonstrate that in the presence of capital and income risk the impact of taxation on consumption choice as well as on economic growth is ambiguous as it affects the mean as well as the variance of disposable income. We observe that the effects of taxation crucially depend on the degree of risk aversion and on the capital income share. Nevertheless, it is possible to solve for welfare maximizing policies. Compared to the deterministic setting, for the optimal policy design additional conditions have to be met.

ISSN 0949 – 9962
JEL classification: D8, D9, E62, O4
Keywords: endogenous growth, taxation, uncertainty

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1 Introduction

This paper is concerned with the macroeconomic effects of government expenditure and taxation within the context of a stochastic endogenous growth model. In extending a widely–used framework, namely the Barro (1990) model of productive government expenditure, with aggregate productivity shocks, we demonstrate that the introduction of technological uncertainty has interesting implications for optimal fiscal policy.

The multiplicative disturbance is assumed to be a stochastic trend described by a geometric Brownian motion, which is the continuous–time correspondent to a random walk with drift. Under this assumption the aggregate income risk affects the macroeconomic equilibrium in several ways. First, the individual will respond to uncertainty within her optimal intertemporal allocation of consumption and saving. If the agent is sufficiently risk averse she will decrease the rate of consumption in favor of additional savings. This motive for *precautionary savings* can be regarded as self–insurance on capital markets over time and was first discussed by Leland (1968) and Sandmo (1970), or more recently by Kimball (1990), Weil (1993), Xu (1995) and Smith (1996a).

Next, it is well known from modern growth theory that the growth rate of the economy is endogenously determined by the factors that affect aggregate savings. Hence, productivity shocks have to be taken into account to the extent in which they impact on capital accumulation and growth.

Third, the issue of efficacy and optimality of fiscal policy has to be addressed, as we deal with the stochastic version of the Barro (1990) model. From Domar and Musgrave (1944) or Stiglitz (1969) it is well–known that taxation of risky assets may actually increase the demand for these assets. In this context, taxation of incomes in order to provide public goods may result in counterworking effects on the growth rate of the economy: on the one hand — as known from the deterministic setting —, taxation has a distortionary effect and suppresses growth. On the other hand, taxation provides an insurance as the government participates in individual income risk. This effect may be growth encouraging and was lately discussed by Turnovsky (1993) and Smith (1996b).

In the Barro (1990) model of endogenous growth agents neglect the productivity enhancing effect of government expenditures, for instance of public infrastructure. Hence, the competitively chosen growth rate falls short of the socially optimal one. The contributions of Cazzavillan (1997), Corsetti (1997) and Turnovsky (1997) are devoted to the analysis of public spendings and their role in a stochastically growing economy. While Cazzavillan (1997) focuses on externalities of public goods in the production and consumption sectors, Corsetti (1997) stresses the function of government bonds in the determination of optimal fiscal policy. The main aspects of the analysis of Turnovsky (1997) are the effects from stochastic and nonstochastic government purchases and the interaction between risk and congestion. The present approach differs to their work in that we explicitly take labor inputs into account. In terms of Sandmo (1970), agents in the economy are then exposed to two kinds of risk, namely capital risk and income risk. Together with the stochastic increasing returns technology of the Barro model this gives rise to nontrivial implications for the design of the optimal tax–scheme.

The paper is organized as follows. Section 2 develops the model and determines the macroeconomic equilibrium for the decentralized economy. In section 3 the corresponding conditions for the command optimum are derived. Section 4 discusses the macroeconomic effects of a change in the tax rate and addresses the question of optimal fiscal policy. Section 5 concludes. Technical details are relegated to the Appendices A and B.

2 The Model and Individual Optimization in the Decentralized Economy

We assume an economy populated by a continuum $[0,1]$ of identical infinitely–lived individuals who produce a homogeneous good according to the stochastic Cobb–Douglas technology

$$dY(t) = \gamma K(t)^{\alpha} L(t)^{1-\alpha} G(t)^{1-\alpha}\,(dt + dy(t)). \tag{1}$$

$K(t)$ denotes the privately owned capital stock. There is no population growth. Labor input $L(t)$ is supplied inelastically and normalized to unity. The instantaneous output $dY(t)$ is subject to an aggregate multiplicative productivity shock. $dy(t)$ is a serially uncorrelated increment to a standard Wiener process with zero mean and variance $\sigma^2\,dt$. The production function of a typical producer displays constant returns to scale with respect to physical capital and labor. For simplicity, depreciation is neglected. Following Barro (1990) the government provides public services $G(t)$ to individual firms. These services may be identified as expenditures on infrastructure and display the characteristics of a pure public good, i. e., are assumed to be non–rival and non–excludable. Hence, we do not discuss congestion as Barro and Sala-I-Martin (1992) or Turnovsky (1996, 1997). Public spendings impact directly on productivity of physical capital thus assuring ongoing growth. Yet, agents in the decentralized economy misperceive this productivity enhancing effect on the accumulable factor.

According to the productivity shocks generated by the above described continuous–time geometric Brownian motion, the flow of government expenditure is given by

$$dG(t) = G(t)\,(dt + dy(t)). \tag{2}$$

The government imposes a constant tax of $\tau \in (0,1)$ on capital and labor incomes in each time increment. The stochastic process of government revenues is represented by

$$dT(t) = \tau\,dY(t). \tag{3}$$

The agent has two ways of saving income: buying riskless bonds or investing in risky physical capital. The safe asset $B(t)$ offers a sure instantaneous and constant yield i. Bonds are considered to be perpetuities. With payouts reinvested and continuously compounded $B(t)$ follows the differential equation

$$dB(t) = i\,B(t)\,dt. \tag{4}$$

Wealth $W(t)$ is the sum of the holdings of the two assets

$$W(t) = K(t) + B(t). \tag{5}$$

The identical households are characterized by a time–separable utility function in consumption only. E_0 denotes the mathematical expectation conditional on time 0 information and β is the rate of time preference, positive by assumption.

The objective of a typical agent is to select her rate of consumption as well as her portfolio of assets in order to maximize the expected value of lifetime utility according to the following program taking prices and the tax rate as given

$$\max_{C(t),K(t),W(t)} \quad \mathcal{V}(0) = \mathrm{E}_0 \int_0^{\infty} U[C(t)]\, e^{-\beta t}\, dt \tag{6}$$

$$\text{s. t.} \quad dW(t) = [i\,(W(t) - K(t)) + (1-\tau)\,(r\,K(t) + \omega) - C(t)]\,dt + dw \tag{7}$$

with $K(0) = K_0 > 0$ given, r the pre–tax rate of return on physical capital, ω the wage rate, and the stochastic process of wealth given by $dw = (1-\tau)\,(rK(t)+\omega)\,dy$.

Consumption $C(t)$ is assumed to be instantaneously deterministic. The current period utility function $U[C(t)]$ is strictly concave and of the isoelastic form

$$U[C(t)] = \begin{cases} \frac{1}{1-\rho} C(t)^{1-\rho} & \rho > 0,\, \rho \neq 1 \\ \ln C(t) & \rho = 1 \end{cases} . \tag{8}$$

The parameter ρ denotes the Arrow/Pratt–index of constant relative risk aversion.

Let $V[W(t),t]$ represent the maximum feasible level of expected lifetime utility. Positing the time–separable form $V[W(t),t] = e^{-\beta t}\,J(W)$, application of Itô's Lemma leads to the following objective function[1]

$$\max_{C,K,W} \mathcal{L} = U(C)\,e^{-\beta t} + V_t + V_W[i\,(W-K) + (1-\tau)\,(r\,K+\omega) - C] + \frac{1}{2} V_{WW}\sigma_W^2. \tag{9}$$

[1]For notational convenience the (t) part of the variables is omitted.

The optimality conditions with respect to C, K and W are[2]

$$0 = U'(C) - J'(W), \tag{10}$$

$$0 = J'(W)\,[r(1-\tau) - i] + \frac{1}{2}\frac{\partial \sigma_W^2}{\partial K} J''(W), \tag{11}$$

$$0 = J'(W)\,(i-\beta) + J''(W)\,[i(W-K) + (1-\tau)\,(rK+\omega) - C] + \frac{1}{2} J'''(W)\sigma_W^2, \tag{12}$$

with the variance of wealth given by $\sigma_W^2 = \frac{\mathrm{E}\,(dw)^2}{dt}$.

Condition (10) displays the well known result of equalized marginal utility of consumption over time and determines the accumulation process together with (12)[3]. Condition (11) sets up the optimal portfolio choice.

The optimal time paths for consumption and the portfolio choice are functions of the derivatives of the value function and form a stochastic differential equation in $J(W)$. The solution strategy is trial and error, finding a function $J(W)$ that satisfies the optimality conditions. As household behavior is characterized by constant relative risk aversion, by following Merton (1971), we first conjecture that in macroeconomic equilibrium consumption is a constant, time–invariant fraction of wealth.

$$C(W,t) = \mu\, W(t), \tag{13}$$

where μ denotes the marginal propensity to consume out of wealth. Second, we guess that in steady state the portfolio shares are also constant, such that

$$\frac{dW(t)}{W(t)} = \frac{dK(t)}{K(t)} = \frac{dB(t)}{B(t)}. \tag{14}$$

The results from individual optimization can now be employed to determine the market equilibrium. To derive macroeconomic equilibrium conditions, it is necessary to establish the government budget constraint. In every period of time the government runs a balanced budget. If we do not allow for public deficits or surpluses as in Clemens and Soretz (1997, 1998) or Corsetti (1997), government spendings equal revenues out of taxation, such that

$$G\,(dt+dy) = \tau\,(rK+\omega)\,(dt+dy). \tag{15}$$

The aggregate resource constraint of the economy includes instantaneously deterministic consumption and government expenditure as defined in (2). Market clearing requires

$$dK = dY - dG - C\,dt. \tag{16}$$

[2]Furthermore, the transversality condition $\lim_{t\to\infty} \mathrm{E}_t\left[J(W)e^{-\beta t}\right] = 0$ has to be satisfied to assure that lifetime utility is bounded.

[3]Condition (12) replaces the Bellman equation that is usually used in this kind of setting. Since we are dealing with suboptimal outcomes in the decentralized economy and increasing returns to scale on the aggregate level, Bellman's principle of optimality cannot be applied here.

We assume perfect competition in the factor markets. The factor returns can then be obtained by using the first–order conditions of the firm problem. In the decentralized economy agents ignore the productivity enhancing effect of public expenditures on private inputs. The pre–tax values of the rental rate of physical capital and the wage rate are determined by the usual marginal productivity conditions

$$r = \alpha\gamma K^{\alpha-1} G^{1-\alpha}, \tag{17}$$

$$\omega = (1-\alpha)\gamma K^{\alpha} G^{1-\alpha}. \tag{18}$$

Given market returns, the equilibrium value of public spendings can be derived from (15)

$$G^{*} = (\tau\gamma)^{\frac{1}{\alpha}} K. \tag{19}$$

From (19) it is obvious that the results from the deterministic Barro model extend to the stochastic setting.

With identical individuals net trade in assets will be zero. All agents are affected by the technological disturbance to the same extent. Under the additional assumption of a closed economy, the aggregate stock of physical capital equals aggregate wealth, $K = W$. If we now employ the first–order condition (11) together with the conjecture for optimal consumption (13), capital return (17), the wage rate (18), and the equilibrium amount of government expenditures (19) the certainty equivalent of real after–tax capital return becomes

$$i = \alpha\gamma^{\frac{1}{\alpha}} (1-\tau)\, \tau^{\frac{1-\alpha}{\alpha}} \left(1 - \rho\gamma^{\frac{1}{\alpha}} \sigma^2 (1-\tau)\, \tau^{\frac{1-\alpha}{\alpha}}\right). \tag{20}$$

Depending only on the exogenously given parameters of the model, the sure interest rate is constant as assumed above. From (20) follows that the risk–free interest rate and the real rate of return on equity differ to the amount of the risk premium.

Given the functional forms of technology (1) and of instantaneous felicity (8) the first–order conditions (10) and (12) together with the guess (13) can be used to obtain a closed–form solution for the propensity to consume out of wealth

$$\mu = \frac{\beta}{\rho} + (1-\tau)\, \gamma^{\frac{1}{\alpha}} \tau^{\frac{1-\alpha}{\alpha}} \left[\frac{\rho-\alpha}{\rho} + \sigma^2 \gamma^{\frac{1}{\alpha}} (1-\tau)\, \tau^{\frac{1-\alpha}{\alpha}} \left(\alpha - \frac{\rho+1}{2}\right)\right]. \tag{21}$$

Individual consumption is a time–invariant function of the exogenously given parameters of the model. The consumption–wealth ratio is constant in steady state, thus confirming the conjecture stated above.

The relation for optimum consumption (21) together with the market clearing condition (16) can now be applied to solve for capital accumulation. Taking expectations allows for the determination of the stationary growth path. Note, that due to the properties of the underlying stochastic process $\mathrm{E}[dy] = 0$. The expected growth rate of the economy $\psi = \left[\frac{1}{K}\frac{\mathrm{E}(dK)}{dt}\right]$ can be derived as follows

$$\psi = \frac{1}{\rho}\left[\alpha\gamma^{\frac{1}{\alpha}}(1-\tau)\tau^{\frac{1-\alpha}{\alpha}} - \beta\right] + \sigma^2\gamma^{\frac{2}{\alpha}}(1-\tau)^2\tau^{\frac{2(1-\alpha)}{\alpha}}\left(\frac{\rho+1}{2} - \alpha\right). \tag{22}$$

The growth rate of the economy is determined endogenously and depends on the factors that affect aggregate savings. It is the sum of two components. The first equals the growth rate known from the deterministic Barro model. The second reflects individual response to aggregate technological risk. From this can be seen, that current shocks may have long–lasting effects on macroeconomic trend. The expected growth rate will exceed deterministic growth if the agent has a motive for precautionary saving, as was first postulated by Leland (1968) and Sandmo (1970). The sign of the second term in (22) is influenced by the agent's attitude towards risk and by the size of the capital income share α. Hence, in this setting, self–insurance on capital markets via precautionary savings does not only depend on the individual degree of risk aversion. This is due to the fact that in contrast to Corsetti (1997) and Turnovsky (1997) we explicitly included labor inputs in our analysis. In this case the individual does not only face risky capital returns but is also exposed to an income risk. Since we considered labor to be inelastically supplied, there are no transitional dynamics and the economy immediately enters steady state growth.

3 Command Optimum

In the decentralized economy the agents neglect the productivity enhancing effect of government expenditure. Only the private marginal product of capital enters into the determination of optimal individual investment and there is a wedge between private and social returns. Consider now a benevolent social planner who maximizes the representative agent's welfare according to (6) while taking account of this distortion.

Under this condition the equilibrium values of the key macroeconomic variables can be derived as follows[4]. The certainty equivalent of risky capital return becomes

$$i^* = (1-\tau)\gamma^{\frac{1}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}\left(1 - \rho\sigma^2\gamma^{\frac{1}{\alpha}}(1-\tau)\tau^{\frac{1-\alpha}{\alpha}}\right). \tag{23}$$

The propensity to consume out of capital is given by

$$\mu^* = \frac{\beta}{\rho} + \frac{\rho-1}{\rho}(1-\tau)\gamma^{\frac{1}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}\left[1 - \frac{1}{2}\rho\sigma^2\gamma^{\frac{1}{\alpha}}(1-\tau)\tau^{\frac{1-\alpha}{\alpha}}\right], \tag{24}$$

[4]The determination of the optimal values is provided in Appendix A.

and the expected growth rate of the economy can be determined as

$$\psi^* = \frac{1}{\rho}\left[(1-\tau)\,\gamma^{\frac{1}{\alpha}}\,\tau^{\frac{1-\alpha}{\alpha}} - \beta\right] + \frac{1}{2}\sigma^2\gamma^{\frac{1}{\alpha}}\,(\rho-1)\,(1-\tau)^2\,\tau^{\frac{2(1-\alpha)}{\alpha}}. \tag{25}$$

A comparison between (22) and (25) shows that it is by no means obvious that the stochastic setting displays the same characteristics as the deterministic Barro model. The growth rate determined by the social planner only exceeds the competitively chosen one if the following condition holds:

Proposition 1 *The expected growth rate of the command optimum is larger than the expected growth rate of the decentralized economy only if the certainty equivalent to capital return is positive*

$$\psi^* \gtrless \psi \quad \Longleftrightarrow \quad 1 > \rho\,\sigma^2\gamma^{\frac{1}{\alpha}}\,(1-\tau)\,\tau^{\frac{1-\alpha}{\alpha}}. \tag{26}$$

For feasible solutions of the model there is a restriction either on the size of the variance of the technological disturbance or on the measure of relative risk aversion[5].

4 Macroeconomic Effects from Tax Policy

The macroeconomic equilibrium of the decentralized economy determined in the previous section now allows for the analysis of tax incidence. The main objective is to examine whether the stochastic Barro model replicates the implications for (optimal) tax policy known from the deterministic setting. The equilibrium values of the expected growth rate, the propensity to consume, and the sure interest rate form a system that completely describes the equilibrium allocation for a given set of fiscal intervention. Taxation affects these macroeconomic variables twofold. Consider the following definitions: $f_1(\tau) = 1-\tau$ and $f_2(\tau) = \tau^{\frac{1-\alpha}{\alpha}}$. The derivatives with respect to the tax rate are easy to calculate with $f_1'(\tau) < 0$ and $f_2'(\tau) > 0$. For a graphical illustration see figure 1. The first, $f_1(\tau)$, stands for the distortionary effect of taxation on accumulation. Here, a rise in the tax rate usually goes along with a decrease in after–tax capital return, thus discouraging growth. The second, $f_2(\tau)$, represents the positive effect of an increase in the government spendings share G/Y on marginal productivity of physical capital that usually is accompanied by an increase in accumulation.

However, from the equilibrium values of the macroeconomic variables we learn that the usually observed effects just hold for the *deterministic* parts. If we look at the diffusion components it is obvious that the growth and consumption effects of a change in

[5]It can be shown that this proposition is also necessary for the transversality condition to be met. An equivalent result can be derived for the stochastic version of the Arrow–Romer (1986) model, cf. Clemens (1997).

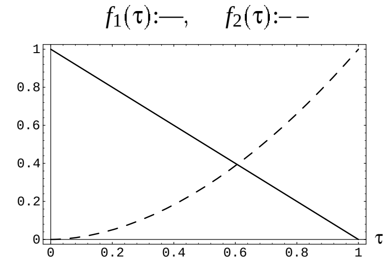

Figure 1: Contrary effects of taxation

$f_1(\tau)$, $f_2(\tau)$ crucially depend on the size of the degree of relative risk aversion and of the capital income share.

In general, the agents respond to uncertainty with a change in their intertemporal consumption–savings decision. Due to the productivity shock future labor and capital income flows are random. As can be seen from the definition of the growth rate (20) second–order effects from the variance of the productivity shock have to be taken into account. The second — *stochastic* — term is positive if the agent has a motive for precautionary savings. In this case $\frac{1}{2}(\rho+1) > \alpha$, that is, the intertemporal income effect dominates the intertemporal substitution effect. For high degree of risk aversion the agent dislikes deviations from a uniform pattern of consumption over time. He tries to smooth consumption flows by increasing savings.

Due to Sandmo (1970), a precautionary motive out of (labor) income risk arises with decreasing absolute risk aversion. A constant degree of relative risk aversion ($\rho > 0$) satisfies this condition. In contrast to this, uncertainty in capital incomes only causes precautionary savings for an attitude towards risk sufficiently high ($\rho \geqslant 1$). Because the agents of our model are exposed to income as well as to capital risk, the margin where both intertemporal effects exactly offset, $\frac{1}{2}(\rho+1) = \alpha$, lies below $\rho = 1, \forall\alpha \in [0,1)$, which is the case of logarithmic preferences[6]. In this case the diffusion component vanishes and the risk averse individual behaves like a risk neutral one.

If the third case, $\frac{1}{2}(\rho+1) < \alpha$, occurs, the intertemporal substitution effect outweighs the income effect. This situation is accompanied by a situation where growth of the stochastic economy even falls below growth of the deterministic scenario. The risk averse individual responds to the aggregate productivity shocks with an increase in consumption above the certainty level.

Following Campbell (1996) empirical evidence suggests a relatively high coefficient of relative risk aversion. For this reason we will restrict our analysis to precautionary

[6]This result makes plain the difference between the model developed here and the stochastic AK–model, where only capital risk is considered (Turnovsky 1995, Ch. 14).

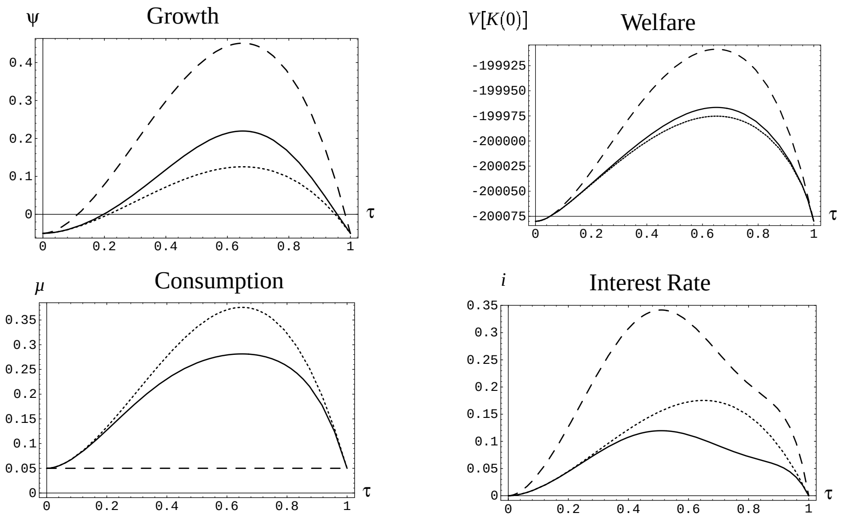

Figure 2: Macroeconomic effects of a change in the tax rate, $\rho = 1$

saving, which is definitely given for any $\rho \geqslant 1$. This argument is supported by Caballero (1990) or Hall and Mishkin (1982) whose empirical findings indicate a strong motive for precautionary savings.

Figure 2 illustrates the differences between the macroeconomic variables of the stochastic and the deterministic setting for logarithmic preferences. The solid line represents the stochastic decentralized economy, the dotted line the corresponding values for the decentralized deterministic setting and the dashed line the stochastic command optimum[7]. From the growth rate and the propensity to consume it becomes obvious that the agent increases savings in favour of a smooth consumption pattern over time.

Let us now turn to the macroeconomic effects of a change in the tax rate. The central point of interest will be the question of optimal, i. e. welfare maximizing policy. It is necessary to consider lifetime utility of a representative agent as specified by (6) evaluated along the competitively chosen path. According to the closed–form solutions describing macroeconomic equilibrium, individual welfare can be derived as follows:[8,9]

$$V[K(0),0] = \frac{K(0)^{1-\rho}}{1-\rho} \cdot \frac{\mu^{1-\rho}}{\beta - (1-\rho)\cdot\left(\psi - \frac{1}{2}\rho\sigma^2\gamma^{\frac{2}{\alpha}}(1-\tau)^2\tau^{\frac{2(1-\alpha)}{\alpha}}\right)}. \tag{27}$$

From (27) it can be seen that the effect of any tax policy on welfare can be assessed in terms of its impact on the propensity to consume and the expected growth rate of the

[7] The parameter values were set as follows: $\alpha = 0.35$, $\beta = 0.05$, $\gamma = 2$, $\sigma = 0.76$.

[8] For details see Appendix B.

[9] For the denominator $\{N\}$ of (27) the following conditions has to be met in order not to violate the transversality condition: $\{N\} := \beta - (1-\rho)\cdot\left(\psi - \frac{1}{2}\rho\sigma^2\gamma^{\frac{2}{\alpha}}(1-\tau)^2\tau^{\frac{2(1-\alpha)}{\alpha}}\right) > 0$.

economy. Ceteris paribus, welfare increases in both, but additionally the indirect (negative) effect from consumption on growth has to be taken into account. As agents neglect the productivity enhancing effect of public spendings within individual optimization, an optimal policy will then be characterized by a tax–scheme that equates marginal (social) costs of government expenditure to its marginal benefits.

The impact of changes in the income tax rate on the expected growth rate and consumption choice is given by

$$\frac{\partial\psi}{\partial\tau} = \gamma^{\frac{1}{\alpha}}\,\tau^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\tau}{\tau}\cdot\frac{1-\alpha}{\alpha}-1\right)\cdot\left[\frac{\alpha}{\rho}+2\sigma^2\gamma^{\frac{1}{\alpha}}(1-\tau)\,\tau^{\frac{1-\alpha}{\alpha}}\left(\frac{\rho+1}{2}-\alpha\right)\right], \tag{28}$$

$$\frac{\partial\mu}{\partial\tau} = \gamma^{\frac{1}{\alpha}}\,\tau^{\frac{1-\alpha}{\alpha}}\left(\frac{1-\tau}{\tau}\cdot\frac{1-\alpha}{\alpha}-1\right)\cdot\left[\frac{\rho-\alpha}{\rho}+2\sigma^2\gamma^{\frac{1}{\alpha}}(1-\tau)\,\tau^{\frac{1-\alpha}{\alpha}}\left(\alpha-\frac{\rho+1}{2}\right)\right]. \tag{29}$$

From (28) and (29) it can be seen that the results from the deterministic Barro model are not necessarily valid in the stochastic context. For $\tau,\gamma > 0$, the derivatives of the expected growth rate and the propensity to consume with respect to τ are the product of two terms. The first term in brackets equals the derivative known from the nonstochastic case. The second one reflects the stochastic elements originating from the model extension. The optimal values can be found by setting the derivatives to 0. The result is:

Proposition 2 *The expected growth rate of the economy attains optimal values either at*

$$\tau^* = 1-\alpha \tag{30}$$

$$\textit{or} \qquad (1-\tau)\cdot\tau^{\frac{1-\alpha}{\alpha}} = \frac{\alpha}{2\rho\sigma^2\gamma^{\frac{1}{\alpha}}\left(\alpha-\frac{\rho+1}{2}\right)}. \tag{31}$$

Condition (30) corresponds to the results known from the original model. The expected growth rate attains a maximum when the distortionary effect from income taxation, $f_1(\tau)$, exactly offsets the capital productivity enhancing effect of an increase in government expenditures, $f_2(\tau)$. The second condition is a polynomial in α and requires numerical solution.

Equivalent results can be derived for the consumption–capital–ratio:

Proposition 3 *The propensity to consume out of capital attains optimal values either at*

$$\tau^* = 1-\alpha \tag{32}$$

$$\textit{or} \qquad (1-\tau)\cdot\tau^{\frac{1-\alpha}{\alpha}} = \frac{\rho-\alpha}{2\rho\sigma^2\gamma^{\frac{1}{\alpha}}\left(\frac{\rho+1}{2}-\alpha\right)}. \tag{33}$$

An analysis of the second–order conditions $\partial^2\psi/\partial\tau^2$, $\partial^2\mu/\partial\tau^2$ shows that whether an optimal tax rate τ^* represents a local/global maximum, or minimum respectively, crucially

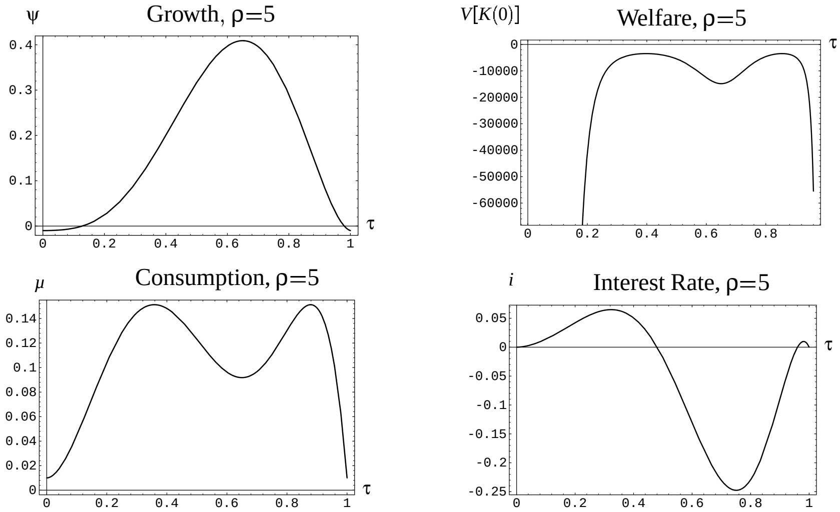

Figure 3: Macroeconomic effects of a change in the tax rate, $\rho = 5$

depends on the the size of the coefficient of relative risk aversion and the capital income share. The second order condition for the growth rate of the competitive economy evaluated at the optimum $\tau^* = 1-\alpha$ can be determined as

$$\left.\frac{\partial^2 \psi}{\partial \tau^2}\right|_{\tau^*=1-\alpha} = -\tau^{\frac{1-\alpha}{\alpha}}\tau^{-1}\gamma^{\frac{1}{\alpha}}\left[\frac{\alpha}{\rho}+\gamma^{\frac{1}{\alpha}}\sigma^2\tau^{\frac{1-\alpha}{\alpha}}\left(\frac{\rho+1}{2}-\alpha\right)\right] \tag{34}$$

From this follows immediately:

Proposition 4 *The growth rate attains a unique maximum at $\tau^* = 1-\alpha$ for the case of $\frac{\rho+1}{2} \geqslant \alpha$, that is the case of certainty equivalence and of precautionary savings.*

Only in the case of precautionary savings and $\tau^* = 1-\alpha$ it is possible to show that the growth rate attains a maximum as the sufficient condition is of negative sign. In other cases the sign remains ambiguous.

As the index of relative risk aversion and the capital income share appear in both terms of the expression in square brackets in (29), the second–order condition for the propensity to consume is of ambiguous sign even when evaluated at $\tau^* = 1-\alpha$.

We conclude our analysis with the derivation of the welfare effect of a change in the tax rate. It is given by

$$\frac{\partial V(0)}{\partial \tau} = K(0)^{1-\rho}\mu^{-\rho}\gamma^{\frac{1}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}\cdot\left(\frac{1-\tau}{\tau}\cdot\frac{1-\alpha}{\alpha}-1\right)\times \tag{35}$$

$$\times\ \frac{\frac{\rho-\alpha}{\rho}+2\gamma^{\frac{1}{\alpha}}\sigma^2(1-\tau)\tau^{\frac{1-\alpha}{\alpha}}\left(\alpha-\frac{\rho+1}{2}\right)-\frac{\mu}{\{N\}}\left[\frac{\alpha}{\rho}+\frac{1}{2}\gamma^{\frac{1}{\alpha}}\sigma^2(1-\tau)\tau^{\frac{1-\alpha}{\alpha}}\left(\frac{1}{2}-\alpha\right)\right]}{\{N\}}$$

The derivative of lifetime utility confirms the results already derived above:

Proposition 5 *The welfare effect of a change in the income tax rate is ambiguous. Expected lifetime utility attains an optimum either at*

$$\tau^* = 1 - \alpha \qquad or \tag{36}$$

$$\frac{\rho-\alpha}{\rho} + 2\gamma^{\frac{1}{\alpha}}\sigma^2(1-\tau)\tau^{\frac{1-\alpha}{\alpha}}\left[\alpha - \frac{\rho+1}{2}\right] = \frac{\mu}{\{N\}}\left[\frac{\alpha}{\rho} + \gamma^{\frac{1}{\alpha}}\sigma^2(1-\tau)\tau^{\frac{1-\alpha}{\alpha}}(1-2\alpha)\right]. \tag{37}$$

Equation (37) again is a polynomial in the tax rate τ and does not allow for an analytical solution[10]. For this reason, we performed some numerical computations. In figure 2 and 3, we contrast two cases which differ only to the underlying size of the coefficient of relative risk aversion. In both cases we assume a motive for precautionary savings, in figure 2 with logarithmic preferences, $\rho = 1$, in figure (3) with a measure of relative risk aversion of $\rho = 5$. The capital income share is set to $\alpha = 0.35$. Let furthermore the rate of time preference be $\beta = 0.05$. The standard deviation of the technological disturbance was set according to Smith (1996b) with $\sigma = 0.76$. Furthermore we assume a low capital coefficient with $\gamma = 2$.

Figure 2 displays the results one would usually expect in the Barro–type framework. All macroeconomic variables have unique maxima in the interval for feasible tax rates, $\tau \in (0,1)$, and for feasible solutions of the model, that is $\psi, \mu, i > 0$. Maximization of the growth rate implies maximization of individual welfare. Contrary to this, in figure 3 the growth rate indeed has a maximum. Yet, this goes along with a local minimum of the propensity to consume and of lifetime utility. This is due to the fact that both, the propensity to consume as well as the expected growth rate impact on welfare and are affected by the change in the tax rate in an ambiguous way. Thus, maximum growth and maximum welfare do not necessarily coincide. This effect does not arise for the Pareto–optimal decision which can easily be verified if we calculate the respective derivative of $V(0)^*$. In this case there is only on unique welfare maximizing tax rate $\tau^* = 1-\alpha$.[11]

However, for a given parameter set, it is important to exclude all tax rates from our analysis that give rise to a negative certainty equivalent of capital return. These situations do not represent feasible consumption programs because the transversality condition is violated.

[10] The corresponding derivative for the planner economy is determined in Appendix C. From (C.1) and (C.2) it can be seen that for a positive certainty equivalent to capital return there is only one unique welfare maximizing tax rate given by $\tau^* = 1-\alpha$, thus reflecting the property of Pareto–optimality.

[11] For the derivation of this result see Appendix C.

5 Conclusions

In this paper we developed a continuous–time stochastic endogenous growth model with a positive production externality due to productive government expenditure. A separate analysis of the drift and the diffusion components of the macroeconomic variables lead to the conclusion that especially in the diffusion component taxation may be either growth enhancing or growth depressing, the results depending entirely on the coefficient of relative risk aversion and on the capital income share. We demonstrated that in contrast to the deterministic Barro model there is no unique growth and welfare maximizing tax rate, as the second–order effects from the technological disturbance have to be taken into account. Yet, one of the optimal tax rates replicates the result known from the deterministic framework. The share of government purchases is chosen optimally if the marginal costs of government spending equal marginal benefits.

Finally, it is important to stress that for feasibility of an intertemporal consumption program, the stochastic economy discussed here has to suffice additional conditions, namely the restriction on the certainty equivalent of capital return.

A Solution to the Planner's Problem

For simplicity, the analysis is restricted to the closed–economy case, where $K = W$. The planner chooses the consumption–capital ratio and savings as to maximize expected lifetime utility (6) of the representative agent subject to the capital accumulation equation

$$dK = \left[\gamma K^{\alpha} L^{1-\alpha} G^{1-\alpha} - G - C\right] dt + \left[\gamma K^{\alpha} L^{1-\alpha} G^{1-\alpha} - G\right] dy. \tag{A.1}$$

He takes the equilibrium value of public spendings (19) into account. Substitution into (A.1) leads to

$$\frac{dK}{K} = \left[\gamma^{\frac{1}{\alpha}} (1-\tau)\, \tau^{\frac{1-\alpha}{\alpha}} - \frac{C}{K}\right] dt + \gamma^{\frac{1}{\alpha}} (1-\tau)\, \tau^{\frac{1-\alpha}{\alpha}}\, dy. \tag{A.2}$$

The individual budget constraint (7) cannot be applied here as there are increasing returns to scale on the aggregate level. Private inputs cannot be paid according to marginal productivity conditions since the Euler adding–up theorem is no longer met.

As before, the value function $V[K(t),t] = e^{-\beta t} J(K)$ represents maximized lifetime utility. Application of Itô's Lemma and taking expectations implies the following objective function:

$$\mathcal{L} = U(C)\, e^{-\beta t} + V_t + V_K[\gamma^{\frac{1}{\alpha}} (1-\tau)\, \tau^{\frac{1-\alpha}{\alpha}} K - C] + \frac{1}{2} V_{KK} \sigma_K^2, \tag{A.3}$$

Taking partial derivatives with respect to C and K leads to the necessary conditions:

$$
\begin{aligned}
0 =& U'(C) - J'(K) && \text{(A.4)}\\
0 =& J'(K)\left[\gamma^{\frac{1}{\alpha}}(1-\tau)\,\tau^{\frac{1-\alpha}{\alpha}} - \beta\right] + J''(K)\,K\left[\gamma^{\frac{1}{\alpha}}(1-\tau)\,\tau^{\frac{1-\alpha}{\alpha}} - \frac{C}{K} + \sigma^2\gamma^{\frac{2}{\alpha}}(1-\tau)^2\,\tau^{\frac{2(1-\alpha)}{\alpha}}\right]\\
&+\frac{1}{2}\sigma^2\gamma^{\frac{2}{\alpha}}\,J'''(K)\,K^2(1-\tau)^2\,\tau^{\frac{2(1-\alpha)}{\alpha}}. && \text{(A.5)}
\end{aligned}
$$

The conjecture for the optimal program is, as before, that the consumption–capital–ratio $\frac{C}{K} = \mu$ is constant in equilibrium. From (A.4) then follows immediately

$$
J'(W) = C^{-\rho}, \quad J''(W) = -\rho\,\mu\,C^{-(\rho+1)}, \quad J'''(W) = \rho(\rho+1)\,\mu^2\,C^{-(\rho+2)}. \qquad \text{(A.6)}
$$

Substitution into (A.5) and solving for μ leads to expression (24) of the text. The expected growth rate of the economy (25) is then obtained by substituting (24) into the capital accumulation equation (A.2) and taking expectations.

B Derivation of Lifetime Utility

B.1 Decentralized Economy

According to (6) lifetime utility is a function of instantaneously deterministic consumption. From substitution of (21) follows

$$
\mathcal{V}(0) = \mathrm{E}_0 \int_0^\infty \frac{\mu^{1-\rho}\cdot K(t)^{1-\rho}}{1-\rho}\cdot e^{-\beta t}\,dt. \qquad \text{(B.1)}
$$

By assumption the stochastic process of production is a geometric Brownian motion. From this follows that the aggregate stock of capital is log–normally distributed. The expected capital stock of time 0 can then be derived as

$$
\begin{aligned}
\mathrm{E}_0\left[K(t)^{1-\rho}\right] =& \exp\left[(1-\rho)\cdot\left(\ln K(0) + \psi t - \frac{1}{2}\sigma_K^2\right) + \frac{1}{2}(1-\rho)^2\sigma_K^2 t\right]\\
=& K(0)^{1-\rho}\cdot e^{\left\{(1-\rho)\cdot\left(\psi - \frac{1}{2}\rho\sigma_K^2\right)t\right\}}. && \text{(B.2)}
\end{aligned}
$$

With the initial capital stock exogenously given, a constant growth rate and propensity to consume, combination of (B.1) and (B.2), and integration leads to the expression (27) for maximized individual welfare

$$
V[K(0),0] = \frac{K(0)^{1-\rho}}{1-\rho}\cdot\frac{\mu^{1-\rho}}{\beta - (1-\rho)\cdot\left(\psi - \frac{1}{2}\rho\sigma_K^2\right)}, \qquad \text{(B.3)}
$$

with the variance of physical capital given by $\sigma_K^2 = \sigma^2\gamma^{\frac{2}{\alpha}}(1-\tau)^2\,\tau^{\frac{2(1-\alpha)}{\alpha}}$.

B.2 Planner Economy

The solution procedure is equal to the one described in the preceding section. By substitution of the consumption–capital–ratio and the initial capital stock of the command economy, welfare can be derived as

$$V[K(0),0]^* = \frac{K(0)^{1-\rho}}{1-\rho} \cdot \frac{(\mu^*)^{1-\rho}}{\beta - (1-\rho)\cdot\left(\psi^* - \frac{1}{2}\rho\sigma_K^2\right)} = \frac{K(0)^{1-\rho}\cdot(\mu^*)^{-\rho}}{1-\rho}. \tag{B.4}$$

C Welfare Maximizing Tax Rate of the Planner Economy

The derivative of (B.4) with respect to the tax rate is given by

$$\frac{\partial V(0)^*}{\partial \tau} = K(0)^{1-\rho}\mu^{-(\rho+1)}\gamma^{\frac{1}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}\left[\frac{1-\tau}{\tau}\cdot\frac{1-\alpha}{\alpha} - 1\right]\cdot\left(1-\rho\gamma^{\frac{1}{\alpha}}\sigma^2(1-\tau)\,\tau^{\frac{1-\alpha}{\alpha}}\right) \tag{C.1}$$

From (C.1) it becomes obvious that the restriction on the certainty equaivalent to capital return is crucial for feasible solutions of the model. There only is a unique global maximum of lifetime utility if the second term in brackets is positive. If we now solve for τ^* the well–known result from the deterministic setting can be obtained

$$\tau^* = 1-\alpha. \tag{C.2}$$

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