The Influence of Heterogeneous Preferences on Asset Prices in an Incomplete Market Model

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Abstract

In this paper, we examine an exchange economy with a financial market composed of three assets: a share of a stock, an European call option written on the stock, and a riskless bond. The financial market is assumed to be incomplete and the option is not a redundant asset. In such a case the construction of a riskless hedge-portfolio to valuate the option is unfeasible and therefore the pricing of the assets becomes a simultaneous valuation problem, nonlinearly depending on the preferences of the agents.

First, the case of homogeneous agents (or, equivalently, of a representative agent) is studied. By means of numerical analysis, it can be found that individual preferences have a major impact on the price relation of the assets, including the price of the option. This stays in contrast to the Black-Scholes analysis, where the option is a redundant asset. A unique price relation exists and no trading takes place.

In the case of heterogeneous agents the price relation of the assets crucially depends on the span of heterogeneity of the preferences. Now, trading takes place. The more risk averse agents buy the bond and sell the share and the option, whereas the less risk averse agents buy the option and the share and sell the riskless bond. More surprisingly we find that the representative asset-pricing-model overprices the riskless bond and underprices the option in relation to our model of heterogeneous agents.

Key words: Asset pricing, Incomplete markets, Option pricing, Heterogeneous agents

JEL classification: C60, D52, G12

1 Introduction

The traditional theory of pricing European-style options, first introduced by Black/Scholes (1973) and extended by Merton (1973), uses an arbitrage-free hedging argument. The valuation of options takes place under the assumption that the underlying asset price follows a given exogenous process. Derivative assets like options are viewed as redundant securities which payoffs can be replicated by portfolios of primary assets. Thus, the market is assumed to be complete without the option. In this approach, preferences of the investors are irrelevant. The only requirement on the preferences is that they are monotonic, such that the possibility of a riskless profit is immediately taken.

In this article we investigate the case of an option as a non-redundant asset. The payoff of the option is unique and cannot be reproduced by a hedge portfolio of a combination of primary assets. A simple economy is considered, where only a riskless bond, shares of a stock and an option written on the stock are available in the financial market. This model is related to the one assumed by Drees/ Eckwert (1995) and is originally based on the paper of Lucas (1978). This Lucas type of models usually assume the existence of a representative agent whose preferences price all assets through his utility maximization.

To derive the Black -Scholes formula in a general equilibrium context, a representative agent with constant relative risk aversion is required (see for example Rubinstein (1976) and Brennan (1979)). That preferences have an effect on the pricing of index options has also been found by Bailey/ Schulz (1989).

In this paper the agents differ in their utility functions. We assume that this function does not fit the aggregation theorem from Rubinstein (1974). The economy is assumed to last only one period[1] and is viewed at two dates, at the beginning and at the end of the period.

Franke/ Stapleton /Subrahmanyam (1999) find that option prices can differ from the value of the Black-Scholes model, if the elasticity of the pricing kernel is declining. This implies that the forward price process is no longer a Brownian motion as assumed by Black-Scholes. The shape of the pricing kernel is determined by the preferences of the agents in a complete market setting. They assume the existence of a representative agent to derive this result. Our paper, in contrast, focuses on the incompleteness of the markets and the impact of heterogenous agents.

Most approaches of asset pricing assume homogeneous agents or, equivalently, the existence of a representative agent. But the occurrence of trading requires the existence of heterogeneity. Franke/ Stapleton /Subrah-

[1] Cox/ Ross/ Rubinstein (1979) have interpreted the Black-Scholes approach in a discrete time framework. This is more related to our investigations than the approach of continuous time.

MANYAM (1998) for example, take into consideration the existence of background risks which differ across agents. They find that agents with low or no background risk sell options, whereas agents with high background risk buy options. LELAND(1980) shows that investors who have average expectations, but whose risk tolerance increases with wealth more rapidly than average, buy options, and so do too investors who have average risk tolerance, but whose expectations of returns are more optimistic than average. DETEMPLE/ SELDEN(1991) have shown that, in an incomplete market setting, primary and derivative assets interact. This interaction increases when heterogeneous agents are introduced. They assume agents who have mean-variance utility and who differ in their expectations with respect to the future variance of the market. Agents who assess high future volatility buy options, whereas those who assess low volatility sell options.

In a recent paper LEISEN/ JUDD (2000) show an economy with agents differing in their preferences and facing different income shocks. They find that the spread of demand is influenced by the heterogeneity of the income shocks and not by the different preferences. They also assume a model with three assets: bonds, stocks and options. Unlike our model the prices of the stocks and bonds are given exogenously. Thus the option is priced in relation to the fixed prices of the other assets. This might be the explanation for the different result concerning the preferences.

In this paper we show that differences in the investors´ preferences have an impact on the relationship between the asset prices and the amount of trading in the market. We do so by first assuming homogeneous agents. It can be found that the option and the bond are not traded, but that there exists an equilibrium price for these assets. Because of the homogeneity of the agents, every deviation from the equilibrium prices induces an immediate demand or supply only on one side of the market. Thus, only one equilibrium price relation between the share, the option and the bond can exist for each kind of homogeneous agent. A change in the preferences of the homogeneous agents leads to a new price relation of the share, the option and the bond.

The next step is to introduce agents who differ in their preferences by having different degrees of risk aversion. Under this assumption trading takes place. The agents with a higher degree of risk aversion sell shares and options and buy the riskless bond. The agents with a lower degree of risk aversion take the opposite position: they buy shares and options and sell bonds.

We find out that the amount of trading and the price of the option grows with increasing divergence in the risk aversion. We use the Arrow-Pratt coefficient of relative risk aversion as an indicator for the different attitudes towards risk.

The paper is organized as follows: In section 2, the theoretical model is introduced and the equilibrium asset prices are derived when all agents are homogeneous. Price relations are simulated given the assumption of isoelastic preferences with alternative constant relative risk aversion. In section 3, heterogeneous agents are introduced. The influence of heterogeneity on the asset prices and on the amount of shares and options traded in the market is shown. In section 4, the main conclusions of the paper are summarized.

2 The model

We consider an exchange economy with only one single perishable good. The economy is populated by n agents. In the first step they are assumed to be homogeneous with respect to their risk aversion. Later in this paper, they are assumed to differ in their attitude towards risk.

The economy is viewed in one period at two dates, the beginning $t = 0$ and the end of the period $t = 1$. The state of nature ω at the second date is uncertain. The number of possible states is limited and can be described by $\omega = 1, ..., \Omega$. We assume that there are more than three states of nature, $\Omega > 3$. There exists an objective probability distribution F^I under an information set I at the beginning of the period, where π^I_ω is the probability of the occurrence of state ω under the information set I. The knowledge of the objective probability distribution F^I is common to all agents. For simplicity we drop the index I for the rest of the paper.

The good is produced by a representative firm. The output of this firm $Y(\omega)$ varies at the end of the period according to the state of nature w. For simplicity the commodity is taken as a numeraire.

The financial market is composed of three divisible assets. This means that the market is incomplete because of the assumption $\Omega > 3$.

First, we have a primary asset which is the stock of the representative firm. The stock gives the owner a claim of a fraction $y(\omega)$ of the representative firm´s production $Y(\omega)$. Thus, the payoff $y(\omega) = (y(1), ..., y(\Omega))$ is a vector in $\mathbb{R}^{\Omega}_{++}$. The stocks are in positive net supply. Each agent is endowed with one share at the beginning of the period.

Second, we have a call option written on the stock, which is in zero net supply. The option is a security, that gives the holder the right, but not the obligation to purchase a share of the underlying stock at a previous fixed strike price k on the maturity date of the option. In our case the maturity date is the end of the period ($t = 1$) and the payoff of the option is:

$$g(\omega) = max(y(\omega) - k; 0) \tag{1}$$

Thus $g(\omega) = (g(1), ..., g(\Omega))$ is a vector in the positive orthant of $\mathbb{R}^{\omega}$.

And third, there exists a riskless asset, a bond, in zero net supply. This bond pays in each possible state of nature one unit of the good.

The financial market opens at the beginning of the period, when the state of nature has not yet been revealed. The agents are able to take long or short positions in the option, the bond and the stock. Thus, each agent

n possesses at $t = 1$ a portfolio x^n composed of the amount of shares $x^{n,s}$, options $x^{n,o}$ and bonds $x^{n,b}$ he has obtained at $t = 0$.

We also assume that at the end of the period, at date t=1, the payoffs of all assets are consumed. The price of the stock, the bond and the option at $t = 0$ is $p = (p^s, p^b, p^o)$. The agent has, under the available information I, a density function $F(p^1(\omega); I)$ about the distribution of the asset prices $p^1(\omega)$ at date $t = 1$. The economy ends at date $t = 1$ and the assets generate only one payoff at this date. Hence, the prices of the assets $p^1(\omega)$ are equal their payoffs at date $t = 1$: $y(\omega)$, $g(\omega)$ and always one for the bond.

We assume that the agents in the economy have a utility function depending on the consumption of the only available good. Consumption takes place only once per period, so that the payoff of agent n´s portfolio $x^n = (x^{s,n}, x^{o,n}, x^{b,n})$ at the end of the period is totally consumed and, thus, equals c^n.

$$c^n(\omega) = y(\omega)x^{s,n} + g(\omega)x^{o,n} + x^{b,n} \tag{2}$$

Because of the zero net supply of the option and the bond, we require for an equilibrium:

$$\sum_{n=1}^{N} c^n(\omega) = Y(\omega) \tag{3}$$

In the following the index n will be dropped for simplicity.

In each state of nature, the agents have a von Neumann Morgenstern utility function $v(c(\omega))$ depending on the payoff of their portfolio, with the following properties:

$$v'(c(\omega)) > 0 \quad v''(c(\omega)) < 0 \quad \lim_{c \to 0} = \infty \tag{4}$$

Hence, the agent´s expected utility $E[v]$ for the end of the period is given by:

$$E[v(c)] = \sum_{\omega=1}^{\Omega} \pi_\omega v(y(\omega)x^s + g(\omega)x^o + x^b) \tag{5}$$

The agent tries to maximize his expected utility by choosing a portfolio at the beginning of the period. He faces the budget constraint, which describes the possibility of trading the initial endowment of one share at date $t = 0$. It is allowed to buy or sell the divisible shares, options and bonds at the market prices $p = (p^s, p^o, p^b)$.

Thus, the maximization problem can be written as follows:

$$\max_{x^s, x^o, x^b} E[v(c)] = \max_{x^s, x^o, x^b} \sum_{\omega=1}^{\Omega} \pi_\omega v(y(\omega)x^s + g(\omega)x^o + x^b) \tag{6}$$

subject to the budget restriction:

$$0 = p^s(x^s - 1) + p^o x^o + p^b x^b \tag{7}$$

The agents´ utility maximization problem is a convex programming one, so that the first order conditions are both, necessary and sufficient, for the existence of a maximum. The first-order conditions for this maximization problem are[2]:

$$\sum_{\omega=1}^{\Omega} \pi_\omega v'(y(\omega)x^s + g(\omega)x^o + x^b)y(\omega) = \lambda p^s \tag{8}$$

$$\sum_{\omega=1}^{\Omega} \pi_\omega v'(y(\omega)x^s + g(\omega)x^o + x^b)g(\omega) = \lambda p^o \tag{9}$$

$$\sum_{\omega=1}^{\Omega} \pi_\omega v'(y(\omega)x^s + g(\omega)x^o + x^b) = \lambda p^b \tag{10}$$

$$p^s(x^s - 1) + p^o x^o + p^b x^b = 0 \tag{11}$$

This leads to the risk-bearing theorem for asset markets:

$$\begin{aligned}&\frac{\sum_{\omega=1}^{\Omega} \pi_\omega v'(y(\omega)x^s + g(\omega)x^o + x^b)y(\omega)}{p^s}\\ &\qquad = \frac{\sum_{\omega=1}^{\Omega} \pi_\omega v'(y(\omega)x^s + g(\omega)x^o + x^b)g(\omega)}{p^o}\\ &\qquad\qquad = \frac{\sum_{\omega=1}^{\Omega} \pi_\omega v'(y(\omega)x^s + g(\omega)x^o + x^b)}{p^b}\end{aligned} \tag{12}$$

To solve for the asset prices, we have to introduce the prices p_ω for the Arrow securities. These securities pay only one unit of good if, and only if, a certain state of nature ω occurs[3]. Thus, (12) becomes:

$$\begin{aligned}&\frac{\sum_{\omega=1}^{\Omega} \pi_\omega v'(y(\omega)x^s + g(\omega)x^o + x^b)y(\omega)}{\sum_{\omega=1}^{\Omega} p_\omega y(\omega)} =\\ &\qquad \frac{\sum_{\omega=1}^{\Omega} \pi_\omega v'(y(\omega)x^s + g(\omega)x^o + x^b)g(\omega)}{\sum_{\omega=1}^{\Omega} p_\omega g(\omega)}\\ &\qquad\qquad = \frac{\sum_{\omega=1}^{\Omega} \pi_\omega v'(y(\omega)x^s + g(\omega)x^o + x^b + x^b)}{\sum_{\omega=1}^{\Omega} p_\omega}\end{aligned} \tag{13}$$

[2] λ is the Lagrangian multiplier.

[3] Because there is no money in this economy, Arrow securities and contingent claims are the same.

(13) is met if the first fundamental theorem of risk-bearing[4] is met for each pair of Arrow securities. This theorem states that, at the investors risk-bearing optimum, the quotient of the marginal utility of the payoffs in different states of nature weighted with the probability of occurrence π_s is equal to the quotient of the Arrow securities in the corresponding states:

$$\frac{\pi_{\omega+1} v'(c(\omega+1))}{\pi_\omega v'(c(\omega))} = \frac{p_{\omega+1}}{p_\omega} \tag{14}$$

Because of (3), the amount of consumption in each state is determined by the payoff of the share, and due to the assumption of all agents being homogeneous in their initial endowment and their preferences, the price of the share at $t = 0$ is the same as its expected payoff:

$$p^s = E[y(\omega)] \tag{15}$$

Hence, it follows that we lose a degree of freedom, because we can determine the price of the share. The securities are priced in relation to the share. The share stands for the productivity of the economy[5].

For simplicity we restrict ourselves to the case of four states of nature, $\Omega = 4$.

Thus, the following system can be solved:

$$\begin{bmatrix} y(1) & y(2) & y(3) & y(\Omega) \\ 1 & -\frac{\pi_1 v'(c(1))}{\pi_2 v'(c(2))} & 0 & 0 \\ 0 & 1 & -\frac{\pi_2 v'(c(2))}{\pi_3 v'(c(3))} & 0 \\ 0 & 0 & 1 & -\frac{\pi_{\Omega-1} v'(c(\Omega-1))}{\pi_\Omega v'(c(\Omega))} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_\Omega \end{bmatrix} = \begin{bmatrix} p^s \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

For the prices of the Arrow-securities[6] we get:

$$p_\omega = \frac{p_s}{\sum_{i=1}^{\Omega} y(\omega) \frac{v'c(i)}{v'c(\omega)}} \tag{17}$$

The prices of the assets are built by adding up the prices of the Arrow securities according to the payoff in the different states of nature:

$$p^s = \sum_{\omega=1}^{\Omega} p_\omega y(\omega) \tag{18}$$

[4]See for example Hirshleifer/ Riley (1992) p. 46.

[5]These assumption implies that the underlying asset is the aggregate national wealth, which is a requirement to obtain the Black-Scholes-formula in discrete time general equilibrium model as stated by Merton(1973) .

[6]See appendix A

$$p^o = \sum_{\omega=1}^{\Omega} p_\omega \max[y(\omega) - k; 0] \tag{19}$$

$$p^b = \sum_{\omega=1}^{\Omega} p_\omega \tag{20}$$

To investigate the influence of preferences on the asset prices $p = (p^s, p^o, p^b)$, we have to specify the utility function. As a benchmark, we choose a utility function of the CRRA (constant relative risk aversion) type:

$$v = \begin{cases} \frac{c^{1-\rho}}{1-\rho} & \rho \neq 1 \\ \ln(c) & \rho = 1 \end{cases} \tag{21}$$

Where ρ is the Arrow-Pratt measure of relative risk aversion[7]. $\rho = 0$ is defined as risk neutrality. For $\rho = 1$ we have the case of logarithmic utility.

So that the agents´ expected utility for the end of the period is given by:

$$E[v] = \sum_{\omega=1}^{\Omega} \pi_\omega \frac{[y(\omega)x^s + g(\omega)x^o + x^b]^{1-\rho}}{1-\rho} \tag{22}$$

First, we investigate the price influence of the risk aversion measure ρ in the homogeneous agent case. If we have the assumptions, that

- the market is incomplete ($\Omega = 4$),
- there exist only three assets,
- all agents have the same CRRA utility function,
- and the wealth is determined by the payoff of the stock,

we find:

$$\frac{\partial p^o}{\partial \rho} < 0 \qquad \frac{\partial p^s}{\partial \rho} = 0 \qquad \frac{\partial p^b}{\partial \rho} > 0 \tag{23}$$

With growing risk aversion the price of the option falls, whereas the price of the bond grows. The price of the share stays the same.

Proof: Because of the assumption that the share is the only asset in positive net supply, it is the only source of aggregate income in the period. Thus, in our model, the price of the share is always its fundamental price, determined by its expected payoff at the end of the period, independently of the level of risk aversion. For the derivation of the price reaction of the two other assets see Appendix B.

[7] The Arrow-Pratt measure is defined as $\rho = -\frac{v''(c(\omega))c(\omega)}{v'(c(\omega))}$.

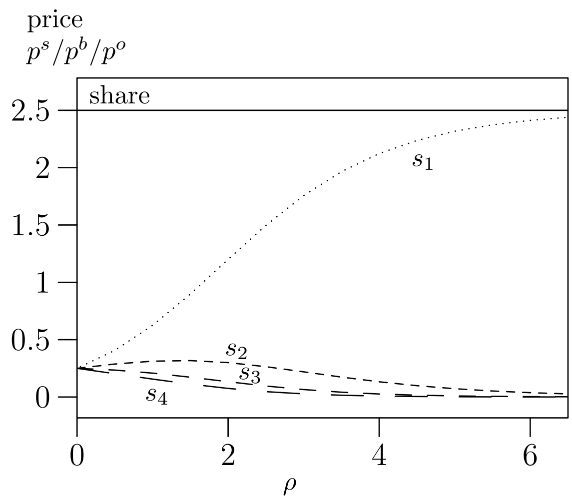

Figure 1: The relationship of the share price (solid line) and the Arrow security prices by varying the relative risk aversion measure ρ. The payoffs in the four states are: $y(\omega) = 1; 2; 3; 4$ $\quad s_1(\omega) = 1; 0; 0; 0$ $\quad s_2(\omega) = 0; 1; 0; 0$ $\quad s_3(\omega) = 0; 0; 1; 0$ $\quad s_4(\omega) = 0; 0; 0; 1$.

Numerical Simulation. To illustrate this findings we use numerical simulations. Computing the prices of the Arrow securities leads to interesting insights of the relationship between the risk aversion and the prices. We can see in figure 1 that in the case of risk-neutrality $\rho = 0$ the prices of all securities are their expected payoffs. The price of the first Arrow security s_1, which has its payoff in state one, grows monotonously with increasing risk aversion. The prices of the other securities decrease monotonously with increasing risk aversion in the region $\rho > 0$.

Moreover, the price increase of the first Arrow security dominates the decrease of the other Arrow securities. Thus, the price of the security which payoff structure includes this Arrow security, the bond, exhibits a growing price with increasing risk aversion. The asset without the first Arrow security in its payoff structure, the option, displays a falling price with higher risk aversion (See figure 2).

Since the bond and the option are in zero net supply, their prices must be such that the agents are indifferent between choosing the stock, the option or the bond. We call these prices of the option and the bond the fair prices for every level of risk aversion of homogeneous agents.

This indifference between choosing any of the assets corresponds with the only possible equilibrium prices. A derivation of this price relation would indicate an immediate entrance of all agents in one side of the market. Thus, in the equilibrium no trading takes place. Trading can only be induced by agents who differ in certain aspects. We will concentrate on differences in the shape of the utility function. Differences in expectations or endowment

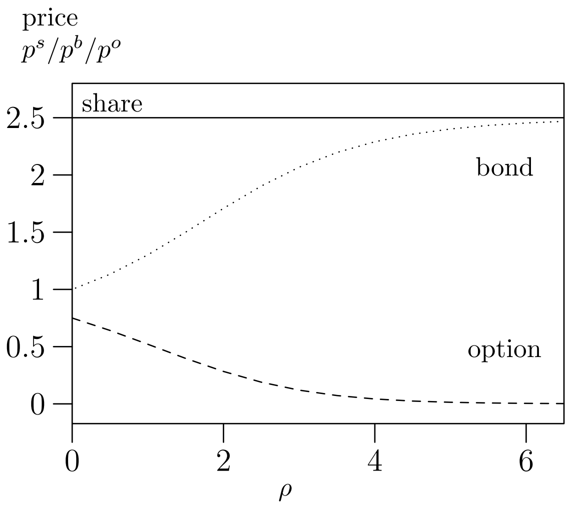

Figure 2: The relationship of the share price (solid line), option price (broken line) and the riskless bond (dotted line) by varying the relative risk aversion measure ρ. The payoffs in the four states are: $y(\omega) = 1; 2; 3; 4 \quad g(\omega) = 0; 0; 1; 2 \quad b(\omega) = 1; 1; 1; 1$.

will not be studied in this paper.

3 The model with heterogeneous agents

We now investigate the relationship between the constellation of the prices and different degrees of heterogeneity between the agents´ risk aversion in the model.

As it was seen in the previous section, the price relation between the option and the share varies with the degree of risk aversion of the representative agent. According to the solution of the maximization problem (6), each type of agent has a different fair price for the option and the bond in relation to the share price, where he is indifferent between purchasing any of the three assets. In this constellation a question arises: Which price will be built at the market, if there are two agents who have different attitudes towards risk? The rest of the paper aims to answer this question.

We assume the existence of two different agents, who have different attitudes towards risk. Let l be the less risk averse agent and h the more risk averse agent, such that:

$$\rho^l < \rho^h \tag{24}$$

Where ρ is again the Arrow-Pratt index of relative risk aversion[8].

Because of the different degrees of risk aversion, the two types of agents have different fair prices. $p^o_{\rho^h}$ is the price for which the relative less risk averse agent is indifferent between purchasing the share or the option. $p^o_{\rho^l}$ is the price for which the relatively more risk averse agent is indifferent between purchasing the share or the option. Hence it follows:

$$p^o_{\rho^h} > p^o_{\rho^l} \qquad \forall \;\; \rho^h < \rho^l \tag{25}$$

We can state the opposite for the bond prices: $p^b_{\rho^h}$ is the price for which the relative less risk averse agent is indifferent between purchasing the share or the bond and $p^b_{\rho^l}$ is the price for which the relatively more risk averse agent is indifferent. Hence, here follows:

$$p^b_{\rho^h} < p^b_{\rho^l} \qquad \forall \;\; \rho^h < \rho^l \tag{26}$$

We assume a Walrasian auctioneer, who determines the equilibrium price $p^{o,*}$ and $p^{b,*}$ so that all markets clear:

$$x^{s,l} + x^{s,h} = n \quad \vee \quad x^{o,l} + x^{o,h} = 0 \quad \vee \quad x^{b,l} + x^{b,h} = 0 \tag{27}$$

These prices will be between the fair prices of both types of agents:

$$p^o_{\rho^h} > p^{o,*} > p^o_{\rho^l} \tag{28}$$

$$p^b_{\rho^h} < p^{b,*} < p^b_{\rho^l} \tag{29}$$

The expected utility function of both agents are:

$$E[v_l] = \sum_{\omega=1}^{\Omega} \pi_\omega \frac{[y(\omega)x^s_l + g(\omega)x^o_l + x^b_l]^{1-\rho^l}}{1-\rho^l} \tag{30}$$

$$E[v_h] = \sum_{\omega=1}^{\Omega} \pi_\omega \frac{[y(\omega)x^s_h + g(\omega)x^o_h + x^b_l]^{1-\rho^h}}{1-\rho^h} \tag{31}$$

The resulting system is a very complex non-linear problem, so that it is not possible to solve it analytically. For that reason we perform numerical computation[9].

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[8] Thus, we have:

$$-\frac{v_l''(c(\omega))c(\omega)}{v_l'(c(\omega))} < -\frac{v_h''(c(\omega))c(\omega)}{v_h'(c(\omega))} \qquad \forall \; \omega = 1, ..., \Omega$$

[9] The computing procedure is similar to the Negishi method, see for example Judd(1998) p.190

Computation procedure. Now, it is necessary to introduce the restriction, that for each agent in any state bankruptcy is excluded[10]:

$$c^n(\omega) > 0 \qquad \forall \;\; n = l, h \quad \omega = 1, .., \Omega \tag{32}$$

We can show, that the more risk averse agent sells shares and options and buys bonds, whereas the less risk averse agent takes the opposite position, buying shares and options and selling bonds.

In order to compute the prices and to determine the amounts of assets which will be traded, we assume a specific case of a less risk averse agent, agent l, and a more risk averse one, agent h, with CRRA utility functions containing $\rho = 1.5$ and $\rho = 4.5$, respectively.

In the equilibrium the number of shares, options and bonds sold and bought are equal (market clearing condition (27)). To obtain the corresponding equilibrium price of the option $p^{o,*}$ and the bond $p^{b,*}$, we use a computation procedure containing the following steps:

1. **Selection of one agent.** We begin with an agent, for example agent l.

   1.a **Fixing prices of the share, bond and option.** In order to determine the first agent´s demand or supply of shares, options and bonds, a certain price relation p of all assets is selected and set fixed.

   1.b **Changing the quantities of the assets in the portfolio** We vary the composition of the agent l´s portfolio according to the budget constraint (7) successively.

   1.c **Checking the no bankruptcy restriction.** The restriction (32) has to be fulfilled in each state for computational reasons (see above).

   1.d **Determination of the expected utility.** The expected utility (30) of each portfolio is determined.

   1.e **Selection of the maximum possible utility and the corresponding trading volume.** For this price relation we detect the agent´s utility maximizing portfolio, composed of a certain proportion of each asset. Thus, for the price combination we obtain the volume of each asset the agent is willing to trade.

[10] In the former, this restriction is not required, because non of the homogeneous agents sell assets in equilibrium thus no bankruptcy occurs. Here, we require this restriction, because, mathematically, the utility function is defined in the negative region of its argument for certain ρ. Thus, we have to prevent, in our numerical simulation, that the consumption becomes negative in one or more states of nature. In an economic perspective we can not reach a negative consumption in any state, because of $\lim_{c\to 0} = -\infty$.

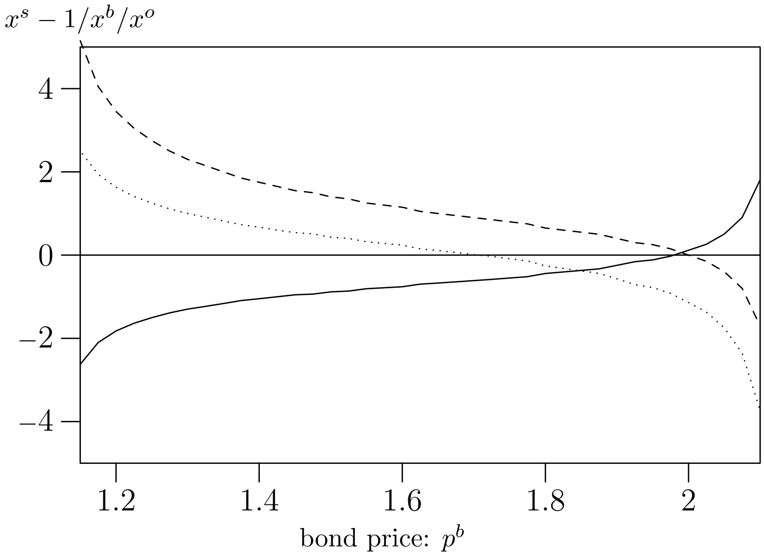

Figure 3: The amount of shares which is demanded by the more risk averse agents ($\rho = 4.5$) is plotted by the solid line, the amount of bonds by the broken line and the amount of options by the dotted line. The option price is set fixed at $p^o = 0,255$. The price of the bond is changed between 1.15 and 2.1.

2. **Changing the bond price.** In a next step, the price of the bond is changed, whereas the option price is still left the same as before and the procedure (a-e) is repeated.

   Thus, we obtain three excess demand functions $f^b_{\rho^l}(p^b)$, $f^s_{\rho^l}(p^b)$, $f^o_{\rho^l}(p^b)$ which describe the amount of assets the agent is willing to trade depending on the price of the bond (figure 3).

3. **Changing the option price.** The price of the option is changed successively and the procedure (1.a-2) is repeated.

4. **Selection of the second agent.** The procedure (1.a-3) is repeated with agent h (figure 4).

5. **Determination of the market clearing prices.** In order to have all markets cleared, we select the price constellation $(p^{o,*}; p^{b,*})$ for which all the resulting demand functions intersect[11].(See figure 5)

[11] To find an equilibrium price vector in this case is not easy. But it is possible by taking sufficiently small steps in the variation of the prices.

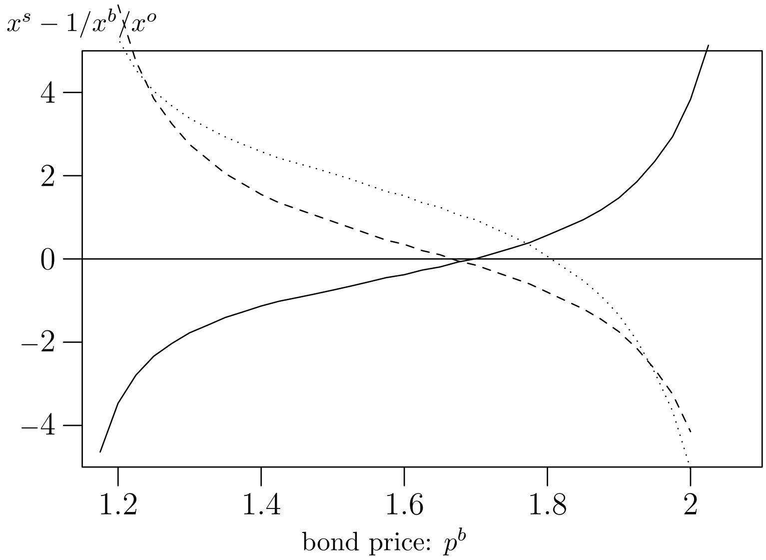

Figure 4: The amount of shares which is demanded by the less risk averse agents ($\rho = 1.5$) is plotted by the solid line, the amount of bonds by the broken line and the amount of options by the dotted line.

$$f^s_{\rho^l}(p^b, p^o) = -f^s_{\rho^h}(p^b, p^o) \wedge \tag{33}$$

$$f^b_{\rho^l}(p^b, p^o) = -f^b_{\rho^h}(p^b, p^o) \wedge \tag{34}$$

$$f^o_{\rho^l}(p^b, p^o) = -f^o_{\rho^h}(p^b, p^o) \tag{35}$$

This procedure ascertains the prices and the corresponding trading volume. As a result, we obtain the information about how many securities are bought and sold by both agents for a certain span of heterogeneity. We can see it in figure 5: the more risk averse agent buys the riskless asset, the bond, and sells the riskier assets, the share and the option. The less risk averse agent takes the opposite position.

We can also determine the equilibrium prices to clear all markets. As we see in figure 5 these prices are 0.255 for the option, 1.78 for the bond and always 2.5 for the share, in the case of one agent having a risk aversion measure of $\rho = 1.5$ and the other one having a $\rho = 4.5$. This prices stay in contrast to the prices which will be built at the market under the existence of a representative agent having a risk aversion measure of $\rho = 3$. If this were the case the price relation for the bond and the option would be of 2.07 and 0.12, respectively (See figure 2). The representative agent model overprices the bond and underprices the option, in relation to the model with heterogeneous agents.

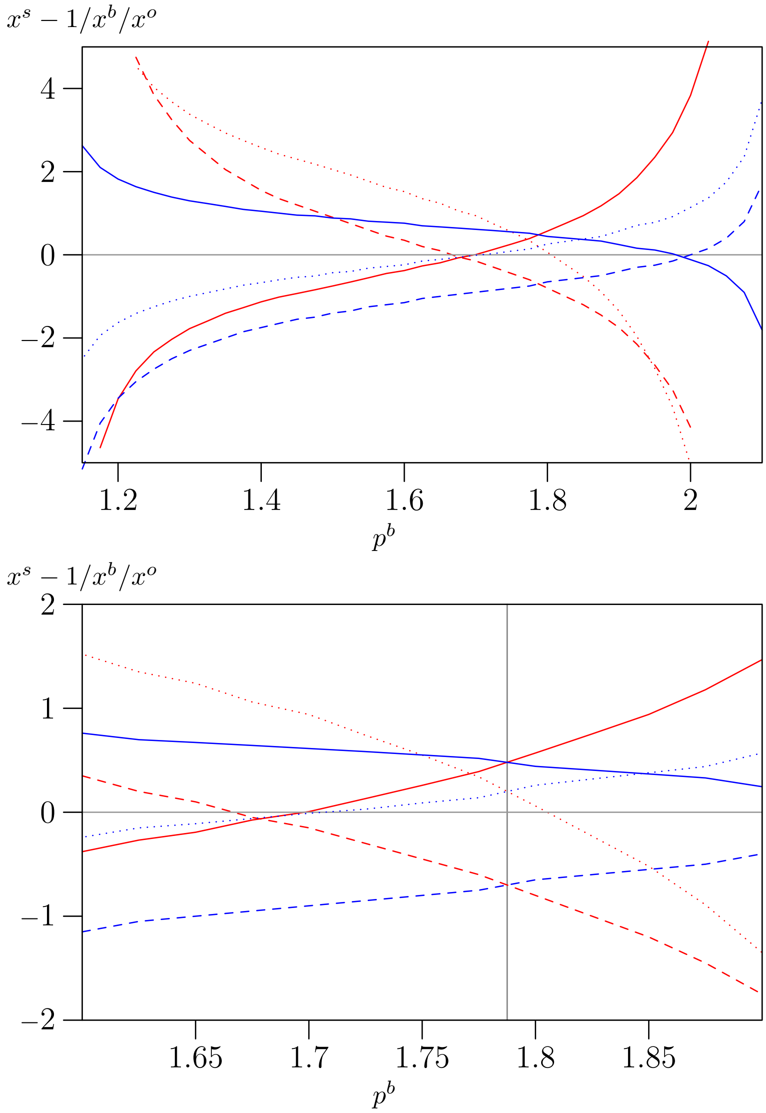

Figure 5: The amount of share is shown by the solid lines, the amount of bonds by the broken line and the amount of options by the dotted line. The red curves belong to the less risk averse agent ($\rho = 1.5$) and the blue ones to the more risk averse one ($\rho = 4.5$). The amount of shares, bonds and options which is demanded by the more risk averse agents is mirrored at the x-axis. The three curves intersect at an option price of 0.255 and a bond price of 1.78. Thus, all markets clear for this price combination. The picture at the bottom is an enlargement of the relevant region in the picture at the top.

The bond price of 1.78 and the option price of 0.255 in this heterogeneous agent model, would indicate a risk aversion measure in the representative agent model of around $\rho = 2.16$.

Altering the span of heterogeneity. In order to understand the influence of heterogeneity, we have repeated the procedure above with different spans of heterogeneity. The span of heterogeneity is in our CRRA utility case the difference between the risk aversion measures ρ. We call this difference Δ:

$$\Delta = \rho^h - \rho^l$$

We start the computation procedure described above, first, with homogeneous agents, who have a relative risk aversion of $\rho_1^h = \rho_1^l = 3$. Then, two agents with different risk aversion coefficients are assumed. Again, the first one is more and the second one is less risk averse than the former homogeneous agents. The whole procedure is repeated with this two heterogeneous agents. In each pass of the procedure, the span of heterogeneity between the agents is heightened. The deviation $|\Delta/2|$ of the Arrow-Pratt risk measure of the homogeneous agents is the same for both agents, as shown:

$$\rho_2^h = \rho_1^h + |\Delta/2| \qquad \rho_2^l = \rho_1^l - |\Delta/2| \tag{36}$$

As a result we obtain the amounts of trading depending on the span of heterogeneity Δ (figure 6) and the corresponding prices of the bond and the option (figure 7).

We find the following relations for the prices:

$$\frac{\partial p^o}{\partial \Delta} > 0 \qquad \frac{\partial p^b}{\partial \Delta} < 0 \tag{37}$$

The equilibrium option price grows with increasing difference, whereas the equilibrium bond price falls. Hence, the heterogeneity of the agents has a significant influence on the equilibrium asset price built in the market. As shown in figures 3 and 4, the higher the risk aversion, the higher the price elasticity of the assets in the relevant area. As a consequence, the less risk averse agent influences the prices of the option and the bond more than the more risk averse one.

For the trading volume we find:

$$\frac{\partial x^s}{\partial \Delta} > 0 \qquad \frac{\partial x^b}{\partial \Delta} > 0 \qquad \frac{\partial x^o}{\partial \Delta} > 0 \tag{38}$$

Growing differences in the heterogeneity of the agents lead to a growing trading volume of the asset. The bond shows the greatest increase of trading volume. The growth of the trading volume of the option declines, so that the quota of the option in the portfolio of the less risk averse agent approaches a certain value by a growing difference in the risk aversion of the two agents.

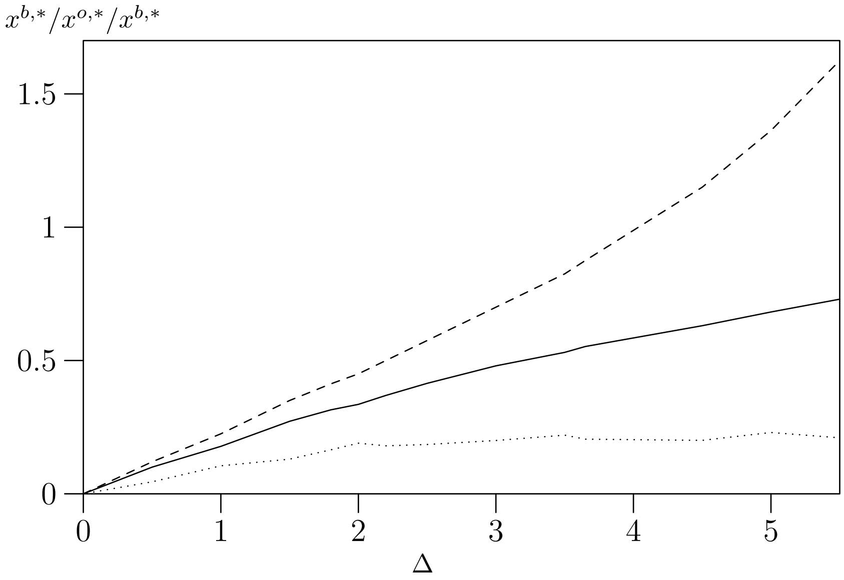

Figure 6: The graph shows the amount of trading of the share $x^{s,*}$ (solid line), the bond $x^{b,*}$ (broken line) and the option $x^{o,*}$ (dotted line) depending on the span of heterogeneity measured by the difference of the Arrow-Pratt measure of relative risk aversion Δ.

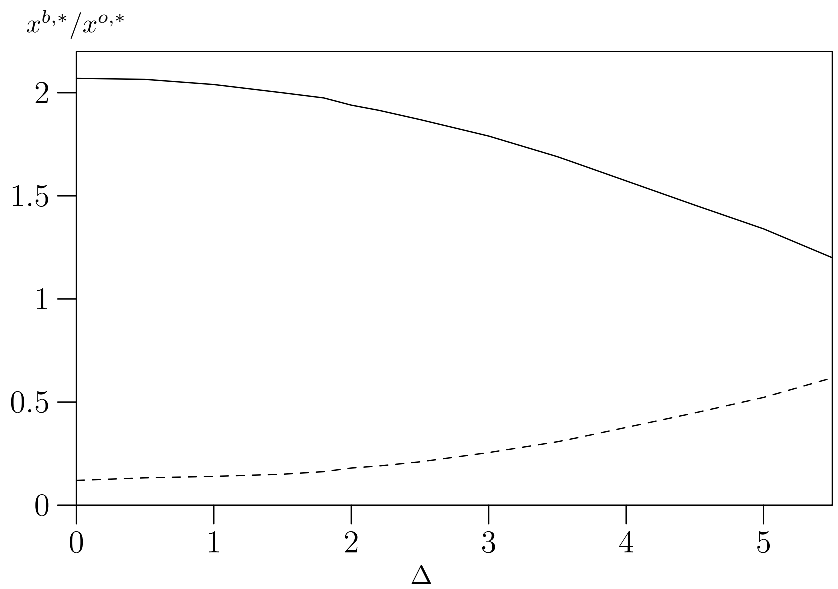

Figure 7: The graph shows the equilibrium price $p^{o,*}$ (broken line) and $p^{b,*}$ (solid line) built in the market depending on the span of heterogeneity measured by the difference of the Arrow-Pratt measure of relative risk aversion Δ and the equilibrium price $p^{o,*}$ built in the market (solid line).

4 Conclusion

This paper studies the influence of preferences on the price relation of different assets. An European-style option, an underlying asset and a riskless bond are assumed in a simple one period equilibrium asset pricing model with an incomplete financial market. It shows, that trading of options and bonds only takes place, when the investors are different. We have investigated the impact of differences in their attitude towards risk. The more risk averse investors will wish to supply the options and demand the bond, whereas the less risk averse investors will wish to take the counter part. It was shown by numerical computations that the span of difference between the agents preferences has a major impact on the bond and option price and, furthermore, on the trading volume. The option price grows with increasing differences between agents, whereas the bond price falls. As a result we find that the assumption of a representative agent, often used in asset pricing models, leads to an undervaluation of the option price and overvaluation of the bond price.

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A Appendix

From the system (16) we get:

$$y(1)p_1 + y(2)p_2 + y(3)p_3 + y(4)p_4 = p^s$$

$$p_1 - \frac{\pi_1 v'(c(1))}{\pi_2 v'(c(2))} p_2 = 0$$

$$p_2 - \frac{\pi_2 v'(c(2))}{\pi_3 v'(c(3))} p_3 = 0$$

$$p_3 - \frac{\pi_3 v'(c(3))}{\pi_4 v'(c(4))} p_4 = 0$$

With the assumption $\pi_1 = \pi_2 = \pi_3 = \pi_4$, this leads to the Arrow security prices :

$$p_1 = \frac{p^s}{y(1) + y(2)\frac{v'(c(2))}{v'(c(1))} + y(3)\frac{v'(c(3))}{v'(c(1))} + y(4)\frac{v'(c(4))}{v'(c(1))}}$$

$$p_2 = \frac{p^s}{y(1)\frac{v'(c(1))}{v'(c(2))} + y(2) + y(3)\frac{v'(c(3))}{v'(c(2))} + y(4)\frac{v'(c(4))}{v'(c(2))}}$$

$$p_3 = \frac{p^s}{y(1)\frac{v'(c(1))}{v'(c(3))} + y(2)\frac{v'(c(2))}{v'(c(3))} + y(3) + y(4)\frac{v'(c(4))}{v'(c(3))}}$$

$$p_4 = \frac{p^s}{y(1)\frac{v'(c(1))}{v'(c(4))} + y(2)\frac{v'(c(2))}{v'(c(4))} + y(3)\frac{v'(c(3))}{v'(c(4))} + y(4)}$$

For the Arrow security ω, the expression can be simplified to:

$$p_\omega = \frac{p^s}{\sum_{i=1}^{\Omega} y(i)\frac{v'c(i)}{v'c(\omega)}} \tag{39}$$

B Appendix

To determine the influence of the risk aversion measure ρ we have to investigate the derivative of each Arrow security price respect to ρ.

In the case of $v'(c) = c^{-\rho}$ this leads to:

$$\frac{\partial p_\omega}{\partial \rho} = -p^s \frac{\sum_{i=1}^{\Omega} y(i)\frac{c(i)^{-\rho}}{c(\omega)^{-\rho}} \ln\left[\frac{c(i)}{c(\omega)}\right]}{\sum_{i=1}^{\Omega} y(i)\frac{c(i)^{-\rho}}{c(\omega)^{-\rho}}}$$

In the case of four states and $y(\omega) = 1; 2; 3; 4$, (39) gives:

$$p_1 = \frac{2.5}{1 + 2\left[\frac{1}{2}\right]^\rho + 3\left[\frac{1}{3}\right]^\rho + 4\left[\frac{1}{4}\right]^\rho}$$

$$p_2 = \frac{2.5}{2^\rho + 2 + 3\left[\frac{2}{3}\right]^\rho + 4\left[\frac{2}{4}\right]^\rho}$$

$$p_3 = \frac{2.5}{3^\rho + 2\left[\frac{3}{2}\right]^\rho + 3 + 4\left[\frac{3}{4}\right]^\rho}$$

$$p_4 = \frac{2.5}{4^\rho + 2^{\rho+1} + 3\left[\frac{4}{3}\right]^\rho + 4}$$

Thus the derivatives are:

$$\frac{\partial p_1}{\partial \rho} = -2.5\frac{-2\frac{1}{2}^\rho \ln(2) - 3\frac{1}{3}^\rho \ln(3) - 4\frac{1}{4}^\rho}{\left(1 + 2\left[\frac{1}{2}\right]^\rho + 3\left[\frac{1}{3}\right]^\rho + 4\left[\frac{1}{4}\right]^\rho\right)^2} > 0 \qquad \forall \quad \rho > 0$$

$$\frac{\partial p_2}{\partial \rho} = -2.5\frac{2^\rho \ln(2) + 3\frac{2}{3}^\rho \ln\left[\frac{2}{3}\right] + 4\frac{1}{2}^\rho \ln\left[\frac{1}{2}\right]}{\left(2^\rho + 2 + 3\left[\frac{2}{3}\right]^\rho + 4\left[\frac{2}{4}\right]^\rho\right)^2} < 0 \qquad \forall \quad \rho > 0$$

$$\frac{\partial p_3}{\partial \rho} = -2.5 \frac{3^\rho \ln(3) + 2\frac{3}{2}^\rho \ln\left[\frac{3}{2}\right] + 4\frac{3}{4}^\rho \ln\left[\frac{3}{4}\right]}{\left(3^\rho + 2\left[\frac{3}{2}\right]^\rho + 3 + 4\left[\frac{3}{4}\right]^\rho\right)^2} < 0 \qquad \forall \quad \rho > 0$$

$$\frac{\partial p_4}{\partial \rho} = -2.5 \frac{4^\rho \ln(4) + 22^\rho \ln(2) + 3\frac{4}{3}^\rho \ln\left[\frac{4}{3}\right]}{\left(4^\rho + 2^{\rho+1} + 3\left[\frac{4}{3}\right]^\rho + 4\right)^2} < 0 \qquad \forall \quad \rho > 0$$

The derivative of the first Arrow security is positive and the others are negative if $\rho > 0$ (See also figure 1). The growth of the first Arrow price dominates all other price decreases. The payoff structure of the bond includes the first Arrow security, thus, its price grows monotonous:

$$\frac{\partial p^b}{\partial \rho} > 0 \qquad \forall \quad \rho > 0$$

The payoff structure of the option, with a strike price of $k = 2$, does not include the first Arrow security, thus, its price decreases with growing ρ:

$$\frac{\partial p^o}{\partial \rho} < 0 \qquad \forall \quad \rho > 0$$

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