Two Notes on Replication in Evolutionary Modeling

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Evolutionary Algorithm Replication and Replicator Dynamics —
An Equivalence Theorem

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Abstract Replicator dynamics and replication as used in evolutionary algorithms are, due to their most basic forms, structurally the same. This short note will prove this thesis.
Although this finding is clear cut and easy to show, it is of great importance for the not yet united families of game theorists on the one hand and evolutionary programmers on the other, meaning that it is perfectly legal and correct to mutually use the tools and findings of each other.
JEL Classification Numbers: C45, C73

Zusammenfassung Replikatordynamik und die Replikation evolutionärer Algorithmen sind in ihren einfachsten Formen strukturell equivalent. Diese kurze Bemerkung beweist die obige Behauptug.
Obwohl die Aussage einfach und leicht zu demonstrieren ist, ist sie nichtsdestoweniger von größter Bedeutung für die noch nicht vereinten Familien der Spieltheoretiker einerseits und Evolutionsprogrammierer andererseits. Das Papier besagt, dass es völlig legal und korrekt ist, sich in der Forschung wechselseitig auf die Methoden und Ergebnisse der jeweils anderen Gruppe zu stützen.

1 Introduction

In modern dynamic economic theory, there are mainly two branches of science claiming to describe evolutionary dynamics, i.e. the process of change in populations of agents or strategies. One is evolutionary game theory making use of various forms of replicator dynamics. The other one is agent based economics and evolutionary programming, essentially using selection/reproduction operators. This paper will point out that at least the two most basic dynamic mechanisms of these branches are exactly the same. It will be comprehensively shown that, due to their mathematical properties, there is no difference between discrete non overlapping generations replicator dynamics and the process of selection and reproduction used in the canonical genetic algorithm.

Evolutionary algorithms (EAs) have often been applied to economic problems in order to serve as a metaphor for learning. Since the work of Axelrod (1987), among these EA models there is a number of models explicitly dealing with game theoretic settings. Moreover, according to Riechmann (2001b), a large number of other EA learning models can be shown to represent evolutionary games, too, even if these models do not look like games at first sight, since they are presented in a 'different language' based on a different scientific background. Just like evolutionary games, EAs provide a population based approach to economic modeling. The most basic and at the same time most often used EA mechanism of transferring one population into the next is the operator of selection and reproduction introduced as a part of the canonical genetic algorithm (CGA, Holland 1975, 1992, Goldberg 1989), which has been characterized in clear mathematical terms by e.g. Davis and Principe (1993) and interpreted as a form of learning by imitation by e.g. Arifovic (1994). And, although the CGA consists of more than this one operator, it can be stripped down to the so called one operator algorithm, which uses selection/reproduction only. This one operator algorithm can be seen as an agent based model of pure simulation learning (Riechmann, 1999).

EA models are essentially agent based simulations of the evolution of the frequencies of agents hosting different strategies in a series of succeeding populations. The populations are always of finite length and in most models even comparably small. This is the reason why in these models, replication, i.e. the selection/reproduction operator, is a biased random process for which only the expected outcome can be calculated.

Replicator dynamics, which are originally based on biologists' ideas about the development of population structures (Maynard Smith, 1982), are the most widespread form of dynamics underlying models of evolutionary population games.[1] The most basic form of replicator dynamics is discrete time replicator dynamics in non overlapping generations models (cf. Samuelson 1997, pp. 63).

Populations underlying replicator dynamics are assumed to consist of infinitely many strategies. (In an alternative interpretation, they are, as Fudenberg and Levine (1998, p. 4) put it, 'continuum populations'.) Thus, although some authors claim that replicator dynamics characterize a random process,

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[1] The name of books and articles focusing replicator dynamics is legion. Some very good ones are Hofbauer and Sigmund (1988, 1998); Weibull (1995); Samuelson (1997); Fudenberg and Levine (1998); Hirshleifer and Martinez Coll (1992).

due to the law of large numbers, the expected outcome is usually seen as being equal to the actual outcome of the process.

This paper will show, that the mathematical properties of CGA replication and the discrete time replicator dynamics in non overlapping generations are structurally equivalent. Although the proof is simple, the result is quite far reaching: It shows that agent based simulations of learning processes based on CGA replication are at least a good first approximation of the evolutionary replicator dynamics.[2] Once the model gets too complex to be treated analytically, i.e. by means of replicator dynamics, an agent based simulation is a very good approximation of the replicator process.

Moreover, for the existing work in both fields of research the result means that it is perfectly justified to compare the outcome of EA simulations and replicator dynamics models: Due to the finding of this paper, EA results are free to be used more or less immediately for further research in evolutionary game theory and vice versa.

2 Equivalence

2.1 The Basics

Imagine a game theoretic population (or a population in an EA) in period t, $\mathbf{P}_t$, which is of length N_t. This means that $\mathbf{P}_t$ consists of N_t agents completely characterized by their individual pure strategies i they plan to play in period t. The set of all possible pure strategies is $\mathcal{S}$.

An example might be

$$\tilde{\mathbf{P}}_t = (A, A, B, C); \quad A, B, C \in \mathcal{S} = \{A, B \ldots Z\}. \tag{1}$$

$\tilde{\mathbf{P}}_t$ has four members, two of them playing strategy A, while one plays B and one C.

Let i abbreviate the strategies (A, B, and C in the example) and $\pi_t(i)$ denote the expected payoff to strategy i in period t.[3] In GA theory, the payoff is often called *absolute fitness*.

In the example, let us assume the payoffs as follows:

$$\pi_t(A) = 2; \quad \pi_t(B) = 3; \quad \pi_t(C) = 3. \tag{2}$$

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[2] Sadly, there is one clear restriction to this result: The similarity does not hold in the ultra long run. For the ultra long run, the one operator algorithm can be shown to always lock in into a state of a homogenous population (Riechmann, 1999), which is not generally the case for replicator dynamics.

[3] In this context, it does not matter, *how* the payoff is gained, i.e. how the game in focus is actually played. For the main question of this paper it is sufficient to know the payoffs.

The absolute frequency of a strategy i in population $\mathbf{P}_t$ or the number of agents playing i in t is given by $N_t(i)$.

For the example, this means

$$N_t(A) = 2; \quad N_t(B) = 1; \quad N_t(C) = 1. \tag{3}$$

The relative frequency of strategy i is given as

$$x_t(i) = \frac{N_t(i)}{N_t}. \tag{4}$$

In the example:

$$x_t(A) = \frac{1}{2}; \quad x_t(B) = \frac{1}{4}; \quad x_t(C) = \frac{1}{4}. \tag{5}$$

Total fitness or aggregate (expected) payoff π_t to the population is the sum of all the individual expected payoffs, i.e.

$$\pi_t = \sum_{i \in S} N_t(i) \, \pi_t(i). \tag{6}$$

For the example:

$$\pi_t = \sum_{i=A}^{Z} N_t(i) \, \pi_i = 2 \cdot 2 + 1 \cdot 3 + 1 \cdot 3 = 10. \tag{7}$$

The average expected payoff is given as

$$\overline{\pi}_t = \sum_{i \in S} x_t(i) \, \pi_t(i). \tag{8}$$

For the exemplary case, the average expected payoff is $\frac{5}{2}$.

It can easily be seen that

$$\frac{\pi_t}{N_t} = \overline{\pi}_t. \tag{9}$$

2.2   Evolutionary Algorithm Replication

The replication process most often applied in EAs is the operator of selection/reproduction stemming from the CGA. This is a process of N_{t+1} times drawing with replacement an agent from population $\mathbf{P}_t$ and copying it into the next population, $\mathbf{P}_{t+1}$. Each agent's probability of being selected in one of the draws equals its relative fitness, which is given by the relation between the agent's fitness and the total fitness of the population, i.e. $\frac{\pi_t(i)}{\pi_t}$. Accordingly, the expected number of agents using strategy i in population $\mathbf{P}_{t+1}$, $N_{t+1}(i)$ is given by the respective agent's relative fitness times the number of agents playing strategy i in the current population times the number of draws, which is the same as the total number of agents in $\mathbf{P}_{t+1}$, N_{t+1}:

$$N_{t+1}(i) = \frac{\pi_t(i)}{\pi_t} N_t(i) N_{t+1}. \tag{10}$$

For the example, this means that[4]

$$N_{t+1}(A) = \frac{2}{10} \cdot 2 \cdot 4 = \frac{16}{10}, \quad N_{t+1}(B) = \frac{12}{10}; \quad N_{t+1}(C) = \frac{12}{10}. \tag{11}$$

Equation (10) is the basic difference equation giving the development of genetic populations in an evolutionary algorithm, consisting of the selection/reproduction operator only. This algorithm has been characterized as modeling a certain form of learning by imitation (Birchenhall, 1995; Riechmann, 1999).

2.3   Replicator Dynamics

In contrast to EA replication dynamics, replicator dynamics are generally characterized in terms of population shares of the strategies in focus. Following Samuelson (1997), the evolution of population shares in discrete non overlapping generations replicator dynamics can be described by

$$x_{t+1}(i) = x_t(i) \frac{\pi_t(i)}{\overline{\pi}_t}. \tag{12}$$

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[4] Note that the resulting *absolute frequencies* are real numbers which will have to be rounded in agent based simulations. The process of rounding, in turn, will cause deviations from the 'true' evolutionary time path of the system modeled. This is the reasons why populations have to be sufficiently large in order to provide a satisfactory approximation of the replicator dynamics process.

For the example, this means that

$$x_{t+1}(A) = \frac{1}{2} \cdot \frac{2}{5/2} = \frac{4}{10}; \quad x_{t+1}(B) = \frac{3}{10}; \quad x_{t+1}(C) = \frac{3}{10}. \qquad (13)$$

Making use of (4), (12) can be transferred into a notation explicitely showing the dynamics in terms of *absolute* frequencies:

$$\frac{N_{t+1}(i)}{N_{t+1}} = \frac{N_t(i)}{N_t} \frac{\pi_t(i)}{\overline{\pi_t}} \qquad (14)$$

$$\Leftrightarrow \quad N_{t+1}(i) = \frac{\pi_t(i)}{\overline{\pi_t}} \frac{N_t(i)}{N_t} N_{t+1}. \qquad (15)$$

Finally, using (9), this results in

$$N_{t+1}(i) = \frac{\pi_t(i)}{\pi_t} N_t(i) N_{t+1}. \qquad (16)$$

Note, that (16) is structurally the same as (10). This means that for population sizes going to infinity, basic EA replication and discrete time replicator dynamics in non overlapping generations models become the same process. Put in other words: EA replication is a good first approximation for replicator dynamics if population sizes are finite.

3   Summary

As this paper is a short one, the summary will be short as well: The only result is the finding that the process of replication in evolutionary algorithms, if modeled by the operator of selection/reproduction from canonical genetic algorithms, and the discrete time replicator dynamics in models of non overlapping generations are, due to their mathematical characteristics, structurally the same. This finding provides a hint for game theorists and GA modelers to mutually take notice of each other and, moreover, to make use of each other's tools and findings.
The equivalence derived in this paper can be extended to cover more than just replication. Processes of noisy replication (as used in form of replicator equations by e.g. Binmore and Samuelson 1994 and Binmore et al. 1995), for example, can be easily 'translated' into EA dynamics and are thus open for further analysis with the help of agent based models.[5]

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[5] For a first step into this direction, see Riechmann (2001a).

Generating Cycles —
A Note on the Use of Elitist Selection Schemes
in State Dependent Problems

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Abstract In evolutionary programming, a family of selection schemes, namely elitist selection might possibly lead to results which are stable cycles even if these results are no sensible solutions to the underlying economic processes. This artefact of elitist selection appears if elitist selection is used in connection with state dependent problems. This note demonstrates this with the help of an example and some generalizations.
JEL Classification Numbers: C45, C73

Zusammenfassung In der evolutionären Programmierung existiert eine Gruppe von Selektionsverfahren, die Verfahren der elitistischen Selektion, die zur Entstehung stabiler Zyklen führen, obwohl solche Zyklen keine „vernünftigen" Lösungen des zugrunde liegenden ökonomischen Modells darstellen. Solche Artefakte elitistischer Selektion entstehen, wenn solche Selektionsverfahren im Zusammenhang mit zustandsabhängigen Problemen angewandt werden. Dieses kurze Papier zeigt dies mit Hilfe eines einfachen Beispiels und einiger Verallgemeinerungen.

1 Introduction

Evolutionary programming deals with the development of a set of micro entities, normally called agents, or genetic individuals, or strategies, in time. These sets of entities are usually called populations. Thus, evolutionary programming is the simulation of the development of populations in time. This development includes a number of 'operators' changing the composition of the population from period to period.
A core operator in every evolutionary program, including a large number of agent based economic models, is the operator of selection. Selection is the part of the evolutionary process which decides on life or death of strategies within populations. In the field of evolutionary programming, there is a number of different such selection schemes. One way of grouping these schemes is differentiating between probabilistic and elitist selection. The main difference between these two families of selection operators is the following:

In elitist selection, the best (i.e. the fittest) member of a number of successive populations is sure to become the member of the new population, i.e. the population of the next period. In probabilistic selection, this is different. Although fitter strategies have a higher probability to become a member of the next generation, every member of a population has a positive (though possibly very small) chance to survive.

In an economic context, especially applied to the problem of boundedly rational agents trying to learn a good solution to an economic problem, both types of selection are often interpreted as a form of learning by imitation.

Most economic problems have another common feature, which is a characteristic of all game theoretic problems. This feature is the state dependency of individual fitness. State dependency means that the fitness of an agent (or strategy, or individual) does not only depend on her own action, but also on what the other members of the population do.

In this note, it will be shown that the use of elitist selection schemes in state dependent problems can result in continuous cycles of macro data, even if the data is far from representing an optimal solution to the underlying economic problem.

2   The Model

The following model will only serve as a simple example and will thus be only briefly described. It is in fact a simple standard cobweb model which is extensively discussed in many other places like Henderson and Quandt (1986). The only special feature of this model is the fact that it is a model with state dependent fitness (i.e. in this case, profit) of the agents involved. This section will point out this feature.

Consider a number of n different firms producing the same perishable good which they have to sell the same period it is produced. The quantity of the good produced by firm i in period t will be denoted by $s_{i,t}$. Every firm i uses the same technology, resulting in cost function (1), where C denotes total production costs. The production function is assumed to have positive marginal costs.

$$C = C\left(s_{i,t}\right) \quad \text{with} \quad \frac{\partial C}{\partial s_{i,t}} > 0. \tag{1}$$

Each firm's fitness is equal to its profit $\pi_{i,t}$, i.e.

$$\pi_{i,t} = s_{i,t}\, p_t - C\left(s_{i,t}\right). \tag{2}$$

Total demand in period t, D_t is given as

$$D_t = \max\left(\gamma - \sigma p_t, 0\right),\tag{3}$$

where γ and σ are positive parameters.
With S_t denoting total supply in t,

$$S_t = \sum_{i=0}^{n-1} s_{i,t},\tag{4}$$

the period t equilibrium price $p_t^\star$ results as

$$p_t^\star = \frac{1}{\sigma}\left(\gamma - S_t\right) \quad \text{for} \quad D_t > 0.\tag{5}$$

Let $S_{-i,t}$ denote the period t aggregate supply of the population except of agent i

$$S_{-i,t} := \sum_{\substack{j=0 \\ j\neq i}}^{n-1} s_{j,t}.\tag{6}$$

This makes it easy to recognize that the equilibrium price depends (among others) on this rest–of–the–population supply $S_{-i,t}$:

$$p_t^\star = \frac{1}{\sigma}\left(\gamma - S_{-i,t} - s_{i,t}\right).\tag{7}$$

Consequently, the profit of each individual i also depends on $S_{-i,t}$.

$$\pi_{i,t} = s_{i,t}\left[\frac{1}{\sigma}\left(\gamma - S_{-i,t} - s_{i,t}\right)\right] - C\left(s_{i,t}\right).\tag{8}$$

The fact that $S_{-i,t}$ appears in the individual profit function (8) proves that the problem is a problem of state dependent fitness. Fitness (i.e. profit) of each individual i depends among others on the sum of individual supplies of all the other members of the population.[1]
For the further course of the paper, one more remark is of some importance: The greater the supply of the rest of the population, $S_{-i,t}$ becomes, the lower is individual i's profit $\pi_{i,t}$:

$$\frac{\partial \pi_{i,t}}{\partial S_{-i,t}} = -\frac{s_{i,t}}{\sigma} \leq 0.\tag{9}$$

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[1] In fact, as the problem is an n–person Cournot game, it is perfectly clear that it is a state dependent problem, because state dependency is known to be the defining characteristic of games (Riechmann, 2001b).

Evolutionary learning models of the above type, using probabilistic selection methods, have been shown to converge to a stable and at least near optimal solution regardless of the question if the underlying parameterization describes a cobweb stable or cobweb unstable setting. (See e.g. Arifovic 1994; Dawid 1996; Franke 1997.)

3   Elitist Selection

It is known that in absence of disturbing forces such as mutation probabilistic selection in the long run inevitably leads to the arise of homogenous populations (Riechmann, 1999). This, in contrast, is not necessarily true for elitist selection.[2]

One type of elitist selection which is used rather often, is the '+'–selection scheme (Michalewicz, 1996). The basic idea is the following: The population in period t, $\mathbf{P}_t$ consisting of n members, is produced by picking the n fittest members of the joint pool of members of the latest two populations, i.e. the populations $\mathbf{P}_{t-1}$ and $\mathbf{P}_{t-2}$.

In order to get a more formalized notation of this operation, let operator $\circ$ stand for the process of elitist reproduction described above.

Thus, the basic law of building a population due to '+'–selection can be written as

$$\mathbf{P}_t = \mathbf{P}_{t-1} \circ \mathbf{P}_{t-2}. \tag{10}$$

In addition, the operator $\circ$ is commutative, i.e.

$$\mathbf{P}_A \circ \mathbf{P}_B = \mathbf{P}_B \circ \mathbf{P}_A. \tag{11}$$

In order to illustrate the finding that this type of selection used in state dependent problems can lead to endlessly ongoing cycles, consider the following populations, which result from an experiment with the cobweb model described above, making use of elitist selection.[3]

The population in period 2 looks as follows (the first column gives the index of the member, the second the quantity produced and the third the fitness).

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[2] For more details on elitist selection see e.g. Michalewicz (1996); Hoffmeister and Bäck (1991); Bäck et al. (1991).

[3] The parameter values used for the simulation were $n = 10$, $\gamma = 100$, $\sigma = 50$, $v = 1.1$. The cost function used is

$$C = C\left(s_{i,t}\right) = F + v\,s_{i,t}^2 \quad \text{with} \quad F, v > 0.$$

```
0       1.8324396068184612      -0.4720862894259463
1       1.8324396068184612      -0.4720862894259463
2       1.4348516579229864      0.2578708819532025
3       1.7619577995937008      -0.31732369164625274
4       1.7619577995937008      -0.31732369164625274
5       1.4348516579229864      0.2578708819532025          P₂
6       1.4348516579229864      0.2578708819532025
7       0.20127731904858503     0.30929305682898
8       0.20127731904858503     0.30929305682898
9       0.20127731904858503     0.30929305682898
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The third population is given as

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0       0.20127731904858503     0.32505925760719134
1       0.20127731904858503     0.32505925760719134
2       0.20127731904858503     0.32505925760719134
3       0.20127731904858503     0.32505925760719134
4       0.20127731904858503     0.32505925760719134
5       1.4348516579229864      0.3702638700588454          P₃
6       1.4348516579229864      0.3702638700588454
7       1.4348516579229864      0.3702638700588454
8       1.4348516579229864      0.3702638700588454
9       1.4348516579229864      0.3702638700588454
```

Due to state dependency, identical strategies yield different fitness values in $\mathbf{P}_2$ and $\mathbf{P}_3$. See e.g. strategy $1.43\ldots$ yielding a fitness of ~ 0.26 in $t = 2$, but of ~ 0.37 in $t = 3$. The reason for this is easy to illustrate: The aggregate supply of population $\mathbf{P}_2$ without strategy no. 6 ($1.43\ldots$) is $S_{-6,2} = 10.662330086$. In the next population, $S_{-6,3} = 6.745793227$. This means that according to (8), because of a decrease in rest–of–the–population supply ($S_{-6,3} < S_{-6,2}$) the profit (and by that: fitness) of strategy $1.43\ldots$ increases: $\pi_{6,3} > \pi_{6,2}$.

By the operation $\mathbf{P}_4 = \mathbf{P}_3 \circ \mathbf{P}_2$, the next population is won:

```
0       0.20127731904858503     0.32505925760719134
1       0.20127731904858503     0.32505925760719134
2       0.20127731904858503     0.32505925760719134
3       0.20127731904858503     0.32505925760719134
4       0.20127731904858503     0.32505925760719134
5       1.4348516579229864      0.3702638700588454          P₄
6       1.4348516579229864      0.3702638700588454
7       1.4348516579229864      0.3702638700588454
8       1.4348516579229864      0.3702638700588454
9       1.4348516579229864      0.3702638700588454
```

A closer look reveals that $\mathbf{P}_4$ is *exactly the same* as $\mathbf{P}_3$. The reason for this is straightforward: Every member of $\mathbf{P}_3$ is better than any member of $\mathbf{P}_2$, such that only the members of $\mathbf{P}_3$ are copied into $\mathbf{P}_4$.

From this, it follows that $\mathbf{P}_5 = \mathbf{P}_4 \circ \mathbf{P}_3$ results as

```
0        1.4348516579229864        0.1932642515283267
1        1.4348516579229864        0.1932642515283267
2        1.4348516579229864        0.1932642515283267
3        1.4348516579229864        0.1932642515283267
4        1.4348516579229864        0.1932642515283267
5        1.4348516579229864        0.1932642515283267        P₅
6        1.4348516579229864        0.1932642515283267
7        1.4348516579229864        0.1932642515283267
8        1.4348516579229864        0.1932642515283267
9        1.4348516579229864        0.1932642515283267
```

$\mathbf{P}_5$ is different from $\mathbf{P}_4 = \mathbf{P}_3$. In this context, it is important to notice that — due to state dependency — the fitness of the only type of agent left has decreased from $\mathbf{P}_4$ to $\mathbf{P}_5$. It has even decreased to such a dramatic extent that again, every member of the population before (i.e. in $\mathbf{P}_4$) has greater fitness than any member of the current population. This certainly means that elitist reproduction will produce a new population $\mathbf{P}_6$ which is identical to $\mathbf{P}_4$:

$$\mathbf{P}_6 = \mathbf{P}_5 \circ \mathbf{P}_4 = \mathbf{P}_4 \, . \tag{12}$$

From this point on, we can compute the whole further process by making use of the basic laws given in (10) and (11):

$$\mathbf{P}_7 = \mathbf{P}_6 \circ \mathbf{P}_5 \quad | \quad \mathbf{P}_6 = \mathbf{P}_4 \tag{13}$$
$$= \mathbf{P}_4 \circ \mathbf{P}_5 \tag{14}$$
$$= \mathbf{P}_5 \circ \mathbf{P}_4 \tag{15}$$
$$= \mathbf{P}_6 \tag{16}$$

$\mathbf{P}_8$ results as

$$\mathbf{P}_8 = \mathbf{P}_7 \circ \mathbf{P}_6 = \mathbf{P}_4 \circ \mathbf{P}_3 = \mathbf{P}_5 \, . \tag{17}$$

Summarizing the facts that $\mathbf{P}_6 = \mathbf{P}_3$, $\mathbf{P}_7 = \mathbf{P}_4 (= \mathbf{P}_3)$, and $\mathbf{P}_8 = \mathbf{P}_5$, it is easy to conclude that the process will result in a lasting and never ending repetition of the sequence of populations $\mathbf{P}_3, \mathbf{P}_3, \mathbf{P}_5$.

Thus, it is obvious that an evolutionary process based on an elitist selection scheme might result in stable cycles which are an artefact of the programming techniques and do not have any support from the underlying (economic) problem.

4   Summary

Elitist selection schemes used in problems with state dependent fitness can in some cases lead to stable endless cycles which are not based on sensible

results of the underlying economic model but are mere artefacts of the type of selection operator used. This result is based on the co working of elitist selection on the one hand and state dependency on the other. If one of these two ingredients is missing, the result will not appear, but if both are present, cyclic simulation results have to be handled with greatest care.

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