Certainty Equivalent in Capital Markets

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Abstract

We generalize the classical concept of a certainty equivalent to a model where an investor can trade on a capital market with several future trading dates. We show that if a riskless asset is traded and the investor has a CARA utility then our generalized certainty equivalent can be evaluated using the sum of discounted one–period certainty equivalents. This is not true if the investor has a HARA utility.

keywords: certainty equivalent, CARA

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1 Introduction

The concept of expected utility (dating back to Bernoulli (1738)) is used in economics to describe the behavior of an investor choosing between several lotteries or alternatives. Its applications include not only portfolio choice but insurance and game theory as well. A certainty equivalent (see Markowitz (1952)) of a risky outcome is a sure–thing lottery which yields the same utility as a random lottery. If the investor is risk–averse the outcome of the certainty equivalent will be less than the expected outcome of the random lottery. A certainty equivalent can be defined in a world where no capital market exists. Our goal is to show how this idea has to be modified if the investor can trade on an incomplete market with several future trading dates. In particular we show that for CARA utility and a capital market with only riskless assets the generalized certainty equivalent can be evaluated using the discounted sum of classical certainty equivalents. The same is not true if the investor has a HARA utility function.

2 A Model with a Riskless Capital Market

To keep our approach as simple as possible we look at three points in time, $t = 0$ (present) and $t = 1, 2$ (future). The future is uncertain. A project realizes cash–flows $CF_0, \widetilde{CF}_1$ and $\widetilde{CF}_2$. Any investor valuing the projects uses his expected utility function $u(x)$. We assume that the utility function is not time–dependent, although there are discount factors δ_1, δ_2. Hence, the utility of an investment is given by

$$u(CF_0) + \delta_1 \mathrm{E}[u(\widetilde{CF}_1)] + \delta_2 \mathrm{E}[u(\widetilde{CF}_2)].$$

We now add a capital market to our model and consider the simplest capital market that is possible: only riskless assets can be traded. Let X_t and Y_t be the amount the investor can put on or get from a riskless money market account. A first idea to generalize a certainty equivalent would be a definition of the form

$$u(C_0 + Y_0) + \delta_1 \cdot \mathrm{E}\left[u(Y_1 - (1+r_f)Y_0)\right] + \delta_2 \cdot \mathrm{E}\left[u(-(1+r_f)Y_1)\right] =_{\mathrm{Def}} \\ u(CF_0 + X_0) + \delta_1 \cdot \mathrm{E}\left[u(\widetilde{CF}_1 + X_1 - (1+r_f)X_0)\right] + \delta_2 \cdot \mathrm{E}\left[u(\widetilde{CF}_2 - (1+r_f)X_1)\right] \tag{1}$$

In the last equation the definition of the certainty equivalent depends on the amount X_t and Y_t which cannot be accepted. A rational investor will optimize her strategy on the capital market: she will choose X_t and Y_t such that both sides on the equation will be as large as possible. Hence

$$\max_{Y_t} u(C_0 + Y_0) + \delta_1 \mathrm{E}\left[u(Y_1 - (1+r_f)Y_0)\right] + \delta_2 \mathrm{E}\left[u(-(1+r_f)Y_1)\right] =_{\mathrm{Def}} \\ \max_{X_t} u(CF_0 + X_0) + \delta_1 \mathrm{E}\left[u(\widetilde{CF}_1 + X_1 - (1+r_f)X_0)\right] + \delta_2 \mathrm{E}\left[u(\widetilde{CF}_2 - (1+r_f)X_1)\right]. \tag{2}$$

This will be our generalization of a certainty equivalent if a capital market is prevalent.

It is evident that riskless payments have to be valued using the riskless interest rate. Although this is obvious it has to be proven using our model (2). To this end we assume that $CF_0 = 0$ and show the following result.

Theorem 1 *If cash-flows are riskless then*

$$C_0 = \frac{CF_1}{1+r_f} + \frac{CF_2}{(1+r_f)^2}.$$

Our proof reveals that this result is not restricted to the time horizon we have choosen. For simplicity we restrict ourselves to $T = 2$. The same will be true if risky assets are traded – if a market price for these risky assets exists our approach will yield the market price.

The utility functions are strictly concave. Hence, X_t^* and Y_t^* are unique. We know that Y_0^* and Y_1^* maximize the left hand side of (2). Substituting

$$\hat{Y}_0 := -\frac{CF_1}{1+r_f} - \frac{CF_2}{(1+r_f)^2} + X_0^*,$$
$$\hat{Y}_1 := -\frac{CF_2}{1+r_f} + X_1^*$$

the right hand side can be written as

$$u\left(\frac{CF_1}{1+r_f} + \frac{CF_2}{(1+r_f)^2} + \hat{Y}_0\right) + \delta_1 \cdot \mathrm{E}\left[u\left(\hat{Y}_1 - (1+r_f)\hat{Y}_0\right)\right] + \delta_2 \cdot \mathrm{E}\left[u\left(-(1+r_f)\hat{Y}_1\right)\right].$$

Since $CF_0 = 0$ this will be the optimal solution of the right hand side of (2) iff

$$C_0 = \frac{CF_1}{1+r_f} + \frac{CF_2}{(1+r_f)^2}$$

holds. But the optimization of the left and the right hand side must yield identical values since the utility functions are strictly concave.

3 CARA–utility

Assume the investor has a utility function of CARA type

$$u(t) = -e^{-at}. \tag{3}$$

Let C_1^B and C_2^B be the (classical *Bernoulli*–)certainty equivalents without a capital market, i.e. riskless payments that yield the same utility as the random cash–flows

$$E\left[u(\widetilde{CF}_1)\right] = u(C_1^B), \qquad E\left[u(\widetilde{CF}_2)\right] = u(C_2^B). \tag{4}$$

We want to clarify the relation between the fair value C_0 and the *Bernoulli*–certainty equivalents. The following theorem holds.

Theorem 2 *If the utility function is of CARA type our certainty equivalent is the riskless discounted* Bernoulli–*certainty equivalent*

$$C_0 = \frac{C_1^B}{1 + r_f} + \frac{C_2^B}{(1 + r_f)^2}.$$

To prove theorem 2 we denote by X_t^* and Y_t^* the optimal money market account from (2). We will furthermore make use of the following two well–known properties of CARA utility for any random variables $\tilde{X}$ and $\tilde{Y}$

$$E\left[u(\tilde{X} + \tilde{Y})\right] = E\left[u(\tilde{X})\right] \cdot E\left[u(\tilde{Y})\right]. \tag{5}$$

If $CF_0 = 0$ equation (2) simplifies to

$$\begin{aligned} &u\left(C_0 + Y_0^*\right) + \delta_1 \cdot E\left[u(Y_1^* - (1 + r_f)Y_0^*)\right] + \delta_2 \cdot E\left[u(-(1 + r_f)Y_1^*)\right] \\ &= u\left(X_0^*\right) + \delta_1 \cdot E\left[u(\widetilde{CF}_1 + X_1^* - (1 + r_f)X_0^*)\right] + \delta_2 \cdot E\left[u(\widetilde{CF}_2 - (1 + r_f)X_1^*)\right], \end{aligned}$$

which using (5) can be written as

$$\begin{aligned} &u\left(C_0 + Y_0^*\right) + \delta_1 \cdot E\left[u(Y_1^* - (1 + r_f)Y_0^*)\right] + \delta_2 \cdot E\left[u(-(1 + r_f)Y_1^*)\right] \\ &\quad = u(X_0^*) + \delta_1 \cdot E\left[u(\widetilde{CF}_1)\right] \cdot E\left[u(X_1^* - (1 + r_f)X_0^*)\right] \\ &\qquad + \delta_2 \cdot E\left[u(\widetilde{CF}_2)\right] \cdot E\left[u(-(1 + r_f)X_1^*)\right]. \end{aligned}$$

With (4) this is equivalent to

$$\begin{aligned} & u\left(C_0+Y_0^*\right)+\delta_1 \cdot u\left(Y_1^*-(1+r_f)Y_0^*\right)+\delta_2 \cdot u\left(-(1+r_f)Y_1^*\right) \\ & = u\left(X_0^*\right)+\delta_1 \cdot u\left(C_1^B\right) \cdot u\left(X_1^*-(1+r_f)X_0^*\right)+\delta_2 \cdot u\left(C_2^B\right) \cdot u\left(-(1+r_f)X_1^*\right) . \end{aligned} \tag{6}$$

We now evaluate the optimal values X_t^* and Y_t^*. We start with Y_0^*. The FOC are

$$\begin{aligned} 0 &= u'\left(C_0+Y_0^*\right)-\delta_1\left(1+r_f\right) u'\left(Y_1^*-(1+r_f)Y_0^*\right) \\ u'\left(C_0+Y_0^*\right) &= \delta_1\left(1+r_f\right) u'\left(Y_1^*-(1+r_f)Y_0^*\right) . \end{aligned}$$

Given our utility functions where $u' = -a\, u$ we get

$$u\left(C_0+Y_0^*\right)=\delta_1\left(1+r_f\right) u\left(Y_1^*-(1+r_f)Y_0^*\right) . \tag{7}$$

Analogously

$$\delta_1\, u\left(Y_1^*-(1+r_f)Y_0^*\right)=\delta_2\left(1+r_f\right) u\left(-(1+r_f)Y_1^*\right). \tag{8}$$

Using a similar approach we arrive at

$$\begin{aligned} 0 &= u'\left(X_0^*\right)-\delta_1\left(1+r_f\right) \mathrm{E}\left[u'(\widetilde{CF}_1+X_1^*-(1+r_f)X_0^*)\right] \\ u'\left(X_0^*\right) &= \delta_1\left(1+r_f\right) \mathrm{E}\left[u'(\widetilde{CF}_1+X_1^*-(1+r_f)X_0^*)\right] \\ u\left(X_0^*\right) &= \delta_1\left(1+r_f\right) \mathrm{E}\left[u(\widetilde{CF}_1+X_1^*-(1+r_f)X_0^*)\right] \\ u\left(X_0^*\right) &= \delta_1\left(1+r_f\right) \mathrm{E}\left[u(\widetilde{CF}_1)\right] \cdot u\left(X_1^*-(1+r_f)X_0^*\right) \\ u\left(X_0^*\right) &= \delta_1\left(1+r_f\right) u\left(C_1^B\right) \cdot u\left(X_1^*-(1+r_f)X_0^*\right). \end{aligned} \tag{9}$$

Finally this gives

$$u\left(C_1^B\right) \cdot u\left(X_1^*-(1+r_f)X_0^*\right)=\delta_2\left(1+r_f\right) u\left(C_2^B\right) \cdot u\left(-(1+r_f)X_1^*\right) \tag{10}$$

We now plug (7) to (10) into (6) such that only $u(X_0^*)$ and $u(C_0 + Y_0^*)$ remain. This gives

$$\left(1 + \frac{1}{1 + r_f} + \frac{1}{(1 + r_f)^2}\right) u(X_0^*) = \left(1 + \frac{1}{1 + r_f} + \frac{1}{(1 + r_f)^2}\right) u(C_0+Y_0^*),$$

and hence

$$u(X_0^*) = u(C_0 + Y_0^*)$$

or, since the utility function is strictly monotonous

$$\begin{aligned} X_0^* &= C_0 + Y_0^* \\ X_0^* - Y_0^* &= C_0 . \end{aligned} \tag{11}$$

Now using (7) to (10) in (6) such that only the terms $u\left(X_1^* - (1 + r_f)X_0^*\right)$ and $u\left(Y_1^* - (1 + r_f)Y_0^*\right)$ left gives

$$\begin{aligned} &\left(1 + r_f + 1 + \frac{1}{1 + r_f}\right) \delta_1\, u\left(Y_1^* - (1 + r_f)\, Y_0^*\right) \\ &\qquad = \left(1 + r_f + 1 + \frac{1}{1 + r_f}\right) \delta_1\, u\left(C_1^B\right) \cdot u\left(X_1^* - (1 + r_f)X_0^*\right) \end{aligned}$$

or (using (5))

$$u\left(Y_1^* - (1 + r_f)Y_0^*\right) = u\left(C_1^B + X_1^* - (1 + r_f)X_0^*\right).$$

This is equivalent to

$$\frac{Y_1^* - X_1^*}{1 + r_f} - (Y_0^* - X_0^*) = \frac{C_1^B}{1 + r_f}. \tag{12}$$

We now use (7) to (10) again in (6) such that only the terms $u\left(-(1 + r_f)X_1^*\right)$ und $u\left(-(1 + r_f)Y_1^*\right)$ remain and we get

$$\begin{aligned} &\left((1 + r_f)^2 + 1 + r_f + 1\right) \delta_2\, u\left(-(1 + r_f)\, Y_1^*\right) \\ &\qquad = \left((1 + r_f)^2 + 1 + r_f + 1\right) \delta_2\, u\left(C_2^B\right) \cdot u\left(-(1 + r_f)X_1^*\right) \end{aligned}$$

or using (5)

$$u\left(-(1+r_f)Y_1^*\right) = u\left(C_2^B - (1+r_f)X_1^*\right).$$

Since the utility function is strictly monotonous

$$X_1^* - Y_1^* = \frac{C_2^B}{1+r_f} \tag{13}$$

and (11) to (13) finish our proof. This proof reveals that the restriction to $T = 2$ is not a necessary restriction.

4 HARA–utility

We are convinced that our theorem 2 is not valid for any utility function. To this end we use a HARA–utility function

$$u(t) = \ln(t) \tag{14}$$

and will show that the riskless discounted *Bernoulli*–certainty equivalent is not the certainty equivalent of our theory. We restrict ourselves to $T = 1$ and only two possible states of nature ω_1 and ω_2. Both states may occur with the same probability. The equation (2) now for $CF_0 = 0$ reads

$$\begin{aligned}
&\ln(C_0 + Y_0^*) + \delta \ln(-(1+r_f)Y_0^*) \\
&= \ln(X_0^*) + \frac{\delta}{2}\left(\ln(CF_1(\omega_1) - (1+r_f)X_0^*) + \ln(CF_1(\omega_2) - (1+r_f)X_0^*)\right).
\end{aligned}$$

The optimal value Y_0^* can easily be evaluated

$$0 = \frac{1}{C_0 + Y_0^*} + \frac{\delta}{Y_0^*} \quad \Longrightarrow \quad Y_0^* = -\frac{C_0}{1+\frac{1}{\delta}}.$$

To get X_0^* we need the FOC on the right hand side

$$0 = \frac{1}{X_0^*} + \frac{\delta}{2}\left(\frac{-(1+r_f)}{CF_1(\omega_1)-(1+r_f)X_0^*} + \frac{-(1+r_f)}{CF_1(\omega_2)-(1+r_f)X_0^*}\right).$$

Some algebraic manipulations lead to

$$X_0^* = \frac{2+\delta}{4(1+r_f)(1+\delta)}\Bigg(CF_1(\omega_1)+CF_1(\omega_2) \pm \sqrt{CF_1^2(\omega_1)+CF_1^2(\omega_2)+2CF_1(\omega_1)CF_1(\omega_2)\frac{\delta^2-4\delta-4}{\delta^2+4\delta+4}}\;\Bigg)$$

Let without loss of generality $CF_1(\omega_1) \le CF_1(\omega_2)$. Then, X_0^* has to be between $\frac{CF_1(\omega_1)}{1+r_f}$ and $\frac{CF_1(\omega_2)}{1+r_f}$. Hence, the solution of the quadratic equation can only be the one with $-$. Now plugging the optimal values in C_0 we get an equation for C_0 that was solved numerically. We also evaluated the riskless discounted *Bernoulli*–certainty equivalent

$$C_1^B = \sqrt{CF_1(\omega_1)\,CF_1(\omega_2)}\,.$$

To show that theorem 2 with a HARA utility does not hold we considered a numerical example. To this end we have choosen $CF_1(\omega_1) = 1$, $r_f = 5\,\%$ and $\delta = 0.95$. A variation of $CF_1(\omega_2) \in (1,7]$ yielded several values for the riskless discounted *Bernoulli*–equivalent and C_0. The figure 1 shows a plot of both values. In our plot both values do not coincide, the difference is larger the larger the volatility of the cash–flow is.

5 Conclusion

We generalized the classical concept of a certainty equivalent to a model where aa riskless capital market with several future trading dates exists.

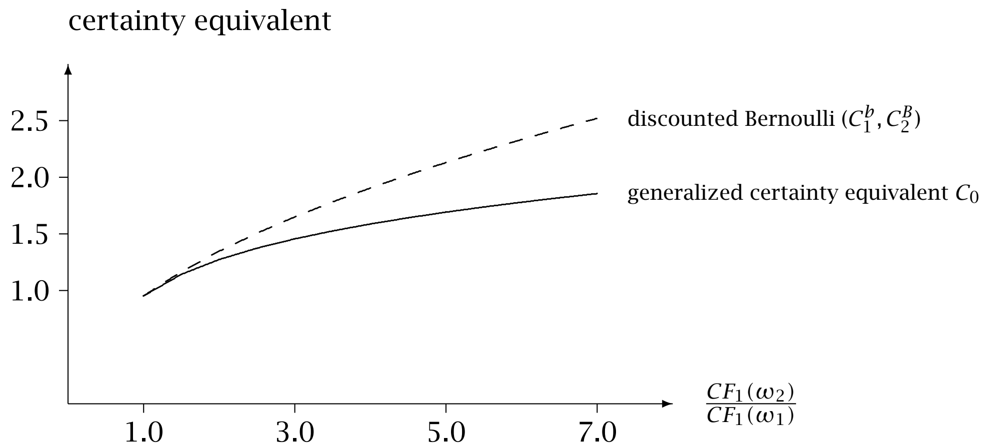

Figure 1: riskless discounted *Bernoulli*–equivalent and depending on volatility (with $r_f = 5\,\%, \delta = 0.95$)

If the investor has a CARA utility then our generalized certainty equivalent can be evaluated using the sum of discounted one–period certainty equivalents. The same is not true with HARA utility.

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