Heterogeneous Preferences and the Representative Investor

Heterogene Präferenzen und der repräsentative Investor

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Abstract

In this paper, I examine an inter-temporal exchange economy with a complete financial market. The economy is populated by two heterogeneous investors who differ from each other in their attitudes towards risk. In such a model, a single representative agent can be created who generates the same asset prices as those generated by the heterogeneous agents.

I analyze the relationship between the preferences of the heterogeneous agents and the preference of the corresponding representative agent and find that the less risk averse agent influences the prices of the contingent claims more than the more risk averse one – even if the more risk averse agent holds most of the contingent consumption in one state of nature.

Key words: Contingent pricing; Heterogeneous agents
JEL classification: G11, G13

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Zusammenfassung

In dieser Arbeit wird eine intertemporale Tauschwirtschaft mit einem vollkommenen Finanzmarkt betrachtet. Es leben zwei Investoren in dieser Wirtschaft, die sich in ihrer Einstellung zum Risiko einer unsicheren Finanzanlage unterscheiden. Es ist bekannt, dass in einem solchen Modell ein einzelner repräsentativer Agent konstruiert werden kann, der durch sein nutzenmaximierendes Verhalten die selben Preise erzeugt, die auch die heterogenen Agenten hervorbringen.

Es wird das Verhältnis zwischen den Präferenzen der heterogenen Agenten und den Präferenzen des dazugehörigen repräsentativen Investors analysiert. Dabei stellt sich heraus, dass der weniger risikoscheu der beiden heterogenen Investoren die Preise der Finanzanlagen stärker beeinflusst als der stärker risikoscheue. Dieses Ergebnis bleibt selbst dann bestehen, wenn der stärker risikoaverse Agent den Anspruch auf die annähernd komplette Auszahlung in einem Naturzustand hält.

1 Introduction

In tradition of LUCAS (1978), most approaches of asset pricing assume homogeneous agents or, equivalently, the existence of a representative agent. But investors are not homogeneous and this could lead to problems in modells using one representative individual [for a discussion see KIRMAN (1992)].

In this paper, I focus on agents who differ in their attitudes toward risk and analyze how they affect equilibrium quantities and prices under different consumption possibilities, and then examine the nature of a representative agent who results in the same prices as would occur with the heterogeneous agents.

It is seen that the representative agent looks more like the less risk averse one, even if the more risk averse agent owns nearly the whole contingent consumption in one state of nature. The preferences of the representative agent display first increasing, then decreasing, relative risk aversion with growing consumption possibilities.

That heterogeneous agents can influence asset prices can be found in several papers, such as LELAND (1980), DETEMPLE/ SELDEN (1991) , and FRANKE/ STAPLETON /SUBRAHMANYAM (1998) . A closely related paper is BENNINGA/MAYSHAR (2000) in which a so-called pricing representative agent is constructed who generates the same prices as an economy with agents differing in their attitudes toward risk. It is found that the pricing representative agent displays decreasing relative risk aversion, leading to certain changes in option prices. In my model, I follow BENNINGA/MAYSHAR

(2000) and assume the same notion of a representative agent. I replace the heterogeneous agents by a representative agent without changing equilibrium prices and aggregate consumption. CONSTANTINIDES (1982) shows it is possible to find such a single composite consumer with preferences that he owns the market portfolio with prices that are determined in the equilibrium under multiple consumers.

In BENNINGA/MAYSHAR (2000) a two-period Arrow-Debreu economy with uncertain consumption possilities in the second period is assumed. The initial endowment is taken as a proxy for the consumption in the first period. It is assumed that the set of states of nature is sufficiently dense, so that every level of positive future aggregate consumption is possible, and it is found that, as output tends to zero, the risk aversion measure of the pricing representative agent looks like the one of the more risk averse agent. When consumption tends to infinity, the pricing representative agent looks like the less risk averse one.

In my analysis, instead of using the initial endowment as a proxy for the consumption in the first state, the exact utility maximizing consumption values are calculated for a finite number of states of nature. This proceeding leads a different result: As consumption tends to zero, the risk aversion measure of the pricing representative agent does not approach the value of the more risk averse one. Thus, the representative Agent allways looks more like the more risk averse agent and the prices of the securities are closer to the prices created by the more risk averse agent being allone at the market.

The paper is organized as follows: In section 2, the theoretical model is introduced and the equilibrium conditions are described. The conditions for the fractions of consumption of both agents are determined in the first period and in each state of nature in the second period. In section 3, the model is solved numerically and the influence of different consumption possibilities is shown. In section 4, the representative investor who generates the same prices as the heterogeneous agents is described. I show that the risk aversion of the representative agent does not become that of the more risk averse agent as consumption tends to zero. Section 5 provides the main conclusions of the paper.

2 The model

We consider a two period Arrow-Debreu economy with a single perishable good which is used as a numeraire to price all financial assets in the economy. The economy is populated by two agents $i = 1, 2$.

The economy is viewed in two periods, the first period, $t = 0, 1$. The state of nature θ in the second period is uncertain. The set of possible states $\Theta = \{1, ..., \theta_n\}$ is finite, hence $\theta \in \Theta$. The aggregate consumption C_0 in the first period is certain and normalized to one: $C_0 = 1$. In the second period $t = 1$, aggregate consumption varies with the state of nature. Knowledge of the objective probability distribution of the occurrence of state θ is common to both agents. Thus, heterogeneous and subjective expectations have no place in our model. The impact of heterogeneous expectations is investigated by DETEMPLE/MURTHY (1994).

A complete financial market opens in the first period, where contingent consumptions for the second period are traded. p_θ stands for the Arrow-Debreu equilibrium prices (state equilibrium prices) for consumption of one unit of the good when state of nature θ realizes. At the beginning of the first period each agent is owns a fraction α_i of the aggregate consumption in the first period as well as a claim of aggregate consumption in each state of nature in the second period. The contingent consumption (contingent claims or Arrow-securities) for the consumption in the second period can be traded between the two agents.

The agents have the following expected utility function:

$$U_i(c_i) = u_i(c_{i,0}) + \beta_i \sum_{\theta=1}^{\theta_n} \pi_\theta u_i(c_{i,\theta}) \tag{1}$$

with

$$u_i(c_i) = \frac{c_i^{1-\rho_i}}{1-\rho_i}. \tag{2}$$

β_i denotes the subjective discount rate and ρ_i the Arrow-Pratt measure of relative risk aversion function of the CRRA utility function of agent i[1]. π_θ is the positive probability of the occurrence of state θ.

Each agent has to choose a consumption program to maximize expected utility under the budget restriction:

$$c_{i,0} + \sum_{\theta=1}^{\theta_n} p_\theta c_{i,\theta} = \alpha_i \left[C_0 + \sum_{\theta=1}^{\theta_n} p_\theta C_\theta \right]. \tag{3}$$

In equilibrium, the marginal rates of substitution for all agents equal the state prices. From the fact that the consumption in the first period is secure – when the agents have to decide the composition of their portfolios of contingent claims – all consumption options in the second period can be valued

[1]The Arrow-Pratt measure is defined as $\rho = -\frac{v''(c(\theta))c(\theta)}{v'(c(\theta))}$. For $\rho = 1$ we have the case of logarithmic utility function.

in relation the marginal utility of the consumption in the first period:

$$p_\theta = \beta_i \pi_\theta \frac{u_i'(c_{i,\theta})}{u_\theta'(c_{i,0})} = \beta_i \pi_\theta \left[\frac{c_{i,\theta}}{c_{i,0}}\right]^{-\rho_i} \quad \forall \quad i, \theta \tag{4}$$

For convex preferences, an initial allocation with positive wealth for each agent, and non zero output in any state, the problem has a unique solution DEBREU (1982).

The price vector $p = (p_1, ..., p_{\theta_n})$ will clear all markets for all contingent claims:

$$\sum_{i=1}^{2} c_{i,\theta} = C_\theta \quad \forall \quad \theta \in \Theta \tag{5}$$

Because of the normalization $C_0 = 1$, the fraction of agent's i consumption in the first period aggregate is equal to his consumption: $c_{i,0}/C_0 = c_{i,0}$. Thus, (4) solved for $c_{i,\theta}$ and inserted in (5) gives

$$\sum_{i=1}^{I} c_{i,0} \left[\frac{\beta_i \pi_\theta}{p_\theta}\right]^{1/\rho_i} = C_\theta \quad \forall \quad \theta \in \Theta. \tag{6}$$

By equation (6) the price vector p is implicitly determined.

2.1 Fraction of consumption in the first period

There is a difference between the consumption of one agent $c_{i,0}$ in the first period and his fraction of the initial wealth α_i. It is shown in appendix A that the relation is given by:

$$\alpha_i = c_{i,0} \left[\frac{1 + \sum_{\theta \in \Theta} p_\theta [\beta_i \pi_{i,\theta}/p_\theta]^{1/\rho_i}}{1 + \sum_{\theta \in \Theta} p_\theta C_\theta}\right]. \tag{7}$$

The relation (7) is central in this model.[2] To understand its relevance,consider the case of two agents having the same initial endowment. To clear the market, in the first period, we have $\alpha_1 + \alpha_2 = c_{1,0} + c_{2,0}$ with $c_{1,0} = c_{2,0} = 0.5 \rightarrow \alpha_1 + \alpha_2 = 1$. Equation (7) differs between the agents only in the following expression for each state of nature θ:

$$[\beta_i \pi_\theta / p_\theta]^{1/\rho_i}. \tag{8}$$

[2]In BENNINGA/MAYSHAR (2000), the initial endowment is taken as a proxy for the consumption in the first period. But I examine the difference between the initial endowment and the consumption in the first period. The influence on the prices in the economy is discussed later.

Assuming both agents have the same time preferences $\beta_1 = \beta_2$, the value of (8) is greater for the less risk averse agent having a smaller risk aversion measure ($\rho_1 < \rho_2$), if the price is sufficiently small, $p_\theta < \frac{1}{\pi\beta}$. In this case $\beta^*\pi_\theta/p_\theta > 1$, and so expression (8) is greater for the bigger exponent $1/\rho_i$, which belongs to the less risk averse agent. If

$$\sum_{\theta=1}^{\theta_n} p_\theta[\beta_1\pi_\theta/p_\theta]^{1/\rho_1} < \sum_{\theta=1}^{\theta_n} p_\theta[\beta_2\pi_\theta/p_\theta]^{1/\rho_2}, \tag{9}$$

the less risk averse agent's consumption share $c_{2,0}$ is smaller than that of the more risk averse one.

Thus, the consumption share of the less risk averse agent in the first period increases with the price of a contingent claim and a decreasing probability of occurrence.

2.2 Fraction of consumption in the second period

Similarly, I can determine the fraction of consumption in each state of nature in the second period – for the derivation see appendix B.

The share of consumption of one agent in an arbitrary state of nature $j \in \Theta$ is given by

$$c_{i,j} = \alpha_i \left[\frac{1 + \sum_{\theta\in\Theta} p_\theta C_\theta}{p_j + \left[\frac{p_j}{\beta^*\pi}\right]^{\frac{1}{\rho_i}} + \sum_{\theta\in\Theta\setminus j} \left[\frac{p_j}{p_\theta}\right]^{\frac{1}{\rho_i}}} \right]. \tag{10}$$

Because of the exponents containing the measure of risk aversion, (10) differs between both agents according to

$$\left[\frac{p_j}{\beta^*\pi}\right]^{\frac{1}{\rho_i}} + \sum_{\theta\in\Theta\setminus j} \left[\frac{p_j}{p_\theta}\right]^{\frac{1}{\rho_i}}. \tag{11}$$

Thus, the crucial point is the price of the examined state of nature in relation to the other prices of the contingent claims. If the other prices are smaller, then $p_j/p_\theta > 1$ and the term $[p_j/p_\theta]^{\frac{1}{\rho_i}}$ is greater for small exponents. The smaller exponent is related to the less risk averse agent. Thus, the share of consumption is greater for the less risk averse agent in the case of small prices. The more risk averse agent has a in greater proportion of expensive contingent claims due to the small consumption possibilities in this state.[3] (This is illustrated in figure 2 in later examples.)

[3]This is also found by Dumas (1989) and Wang (1996) .

3 Numerical simulations

Further study of the prices of the contingent claims must proceed by means of numerical simulations. Because the exponents depend on the risk aversion measure they could have many different values.Thus, this model reduces to a system of non-linear equations, which can be solved by a numerical non-linear equation solver. I have chosen to use a grid laid over the relevant region of portfolio combinations and to determine the equilibrium prices and consumption shares by repeating the search procedure with more refined grids [See JUDD (1998)].

For these computer simulations, I consider the concrete case of an economy with two agents and three possible states of nature in the second period, $\#\Theta = 3$. The less risk averse agent is assumed to have a relative risk aversion coefficient of $\rho_1 = 1$, so that his time separable utility function in each state of nature is logarithmic, the more risk averse agent is assumed to have a risk aversion of $\rho_2 = 7$.[4]

Further parameters have to be fixed: Both agents have the same time discount factor $\beta_1 = \beta_2 = \beta^* = 0.99$. Variables indexed by $*$ belong to the representative agent, introduced in section 4. All states of nature have the same possibility of occurrence $\pi_1 = \pi_2 = \pi_3$. Both agents have the same initial endowment $\alpha_i = 0.5$.[5] Thus, we have the non-linear system to be solved by the numerical solver.

Variation of consumption

To get deeper into the relation between the consumption possibilities in the different states of nature, the heterogeneity and the resulting state prices, the numerical technique is used under changing aggregate consumption to show the influence of different consumption possibilities on the equilibrium quantities and prices.

These exercise is chosen to analyze the behavior of our model under a changing aggregate consumption: $C \to \infty$ and $C \to 0$. In our model with three states we obtain different results from BENNINGA/ MAYSHAR (2000) cf. proposition 3 page 14.

Variation of the consumption in one state.

In this exercise whole aggregate consumption in the first state of nature is

[4] This are the same heterogeneous preferences as assumed in the numerical example of BENNINGA/ MAYSHAR (2000), so it is possible to compare the results.

[5] Because of the market clearing condition (5), it is sufficient to investigate only the budget restriction of one agent. The other agent only takes the contra position and, thus, must have a balanced budget.

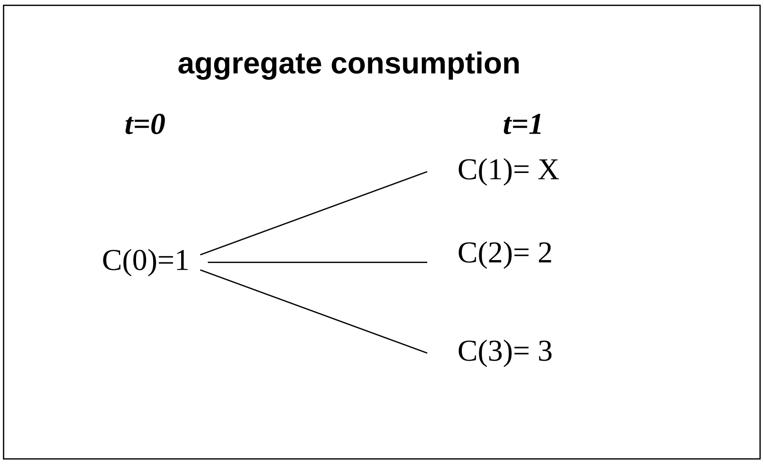

Figure 1: Setup 1. Contribution of consumption over time and states

changed. Consumption in the second and third states is left unchanged, (see figure 1).

Figures 2 and 3 display the results of the numerical simulations. Figure 2 shows the share of aggregate consumption of the less risk averse agent ($\rho_1 = 1$). The densely dotted line represents his share of consumption in the first state of nature ($\theta = 1$). The smaller the aggregate consumption C_1 in state $\theta = 1$, the smaller the consumption of this agent in this state. With an aggregate consumption of 0.1, $c_{1,1}$ nearly approaches zero ($c_{1,1} = 0.016$). The share 1 consumption in the first period (solid line) decreases with C_1, as does the consumption share in both the second (widely dotted) and the third state of nature (dashed).

The second, more risk averse agent, abstains nearly completely from consuming in the state of nature in which the aggregate consumption possibility tends to zero and therefore obtains nearly all consumption in the other states and in the first period. The more risk averse agent tries to even out the consumption over the states and time.

The corresponding contingent prices are shown in figure 3. The first state price (densely dotted line in figure 3) is extremely high with low consumption possibilities. The explanation is intuitive: the state price for a worse state should be higher if consumption has low odds.

Figure 4 provides greater detail for the state prices p_2 and p_3 for the second and third state to illustrate their dependence on different consumption levels in $\theta = 1$. Surprisingly, these prices (dashed line for $\theta = 2$; dotted line for $\theta = 3$) are higher too in case of low consumption possibilities. This is not

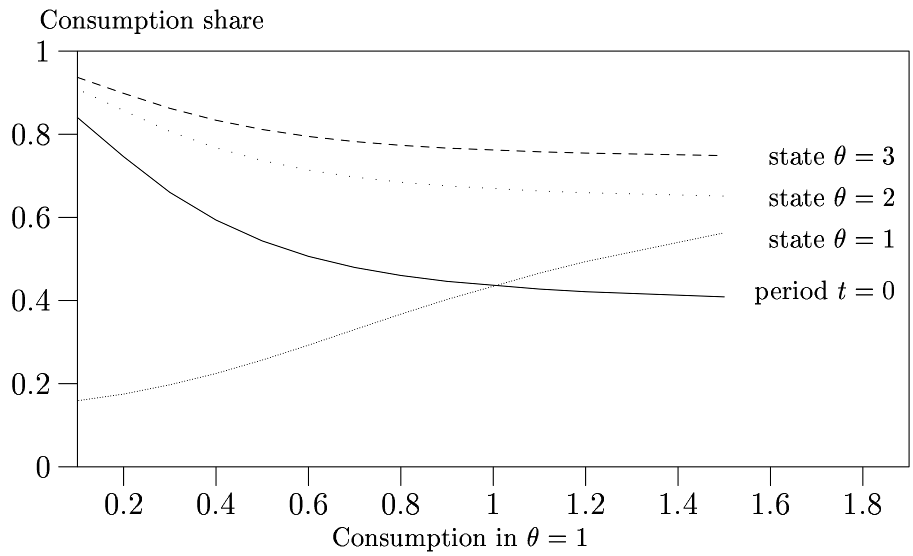

Figure 2: Share of aggregate consumption of the less risk avers agent ($\rho = 1$)

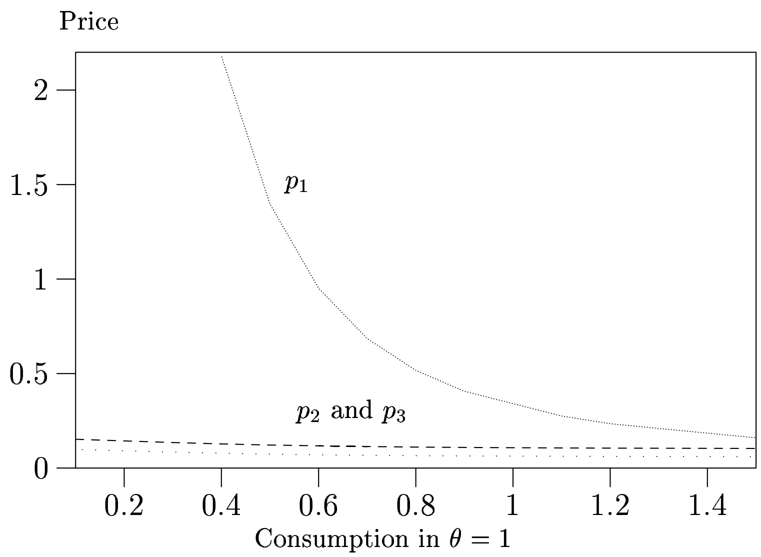

Figure 3: Prices of the contingent claims

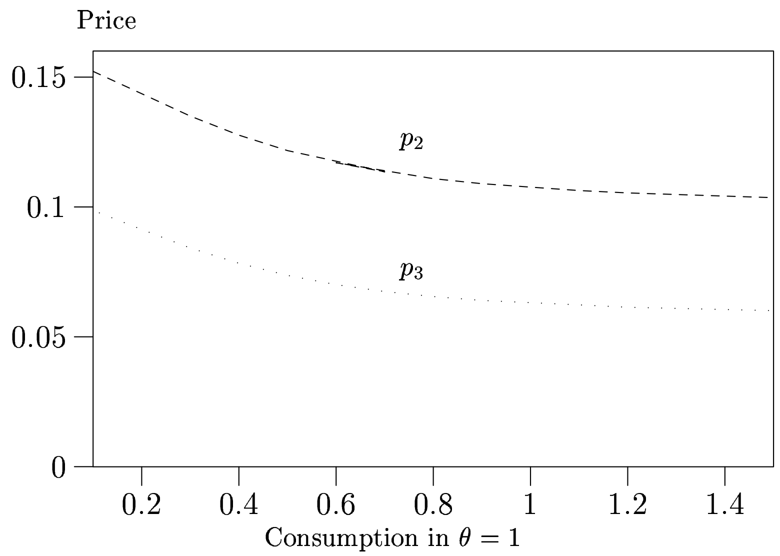

Figure 4: Prices of the second and third contingent claim

directly intuitive because the endowment of consumption in these "good" states of nature stay the same and improve in comparison to the one in the first state. Thus, the prices are not expected to be higher in this case.

4 The pricing representative agent

Next I describe the characteristics of the representative agent. I replace the heterogeneous agents by a representative one without changing equilibrium prices and aggregate consumption. As shown by CONSTANTINIDES (1982), it is possible to find preferences for a single representative investor, who will hold the same market portfolio under the conditions determined by the equilibrium for multiple consumers. In most cases the preferences of the resulting representative investor are not of the same class as those of the heterogeneous agents.[6]

Following again BENNINGA/MAYSHAR (2000), the representative agent has the same utility function as the heterogeneous ones, except that u^* is

[6] A change in the consumption possibilities or endowments normally leads to a different representative agent. As shown by RUBINSTEIN (1974), the class of utility functions, whose aggregation of the preferences is possible, is very restrictive. The CRRA-utilities assumed here, do not belong to this class.

now not of the CRRA type:

$$U^*(C) = u_0^*(C_0) + \beta^* \sum_{\theta=1}^{\theta_n} \pi_\theta u^*(C_\theta). \tag{12}$$

Setting marginal utilities equal to prices, the parameters of this utility function should obey:

$$p_\theta = \beta^* \pi_\theta \frac{u^{*'}(C_\theta)}{u^{*'}(C_0)} \quad \forall \quad \theta. \tag{13}$$

That is, the marginal utility of consumption in the first period relative to that in each possible state of nature in the second period, weighted with the probability of occurrence of the corresponding state of nature, should be the price of a contingent claim in this state of nature.

As shown by BENNINGA/MAYSHAR (2000), the representative rate of time preference β^* is a weighted average of the rates of time preference of the heterogenous agents. The weight of each agent depends on his initial endowment α_n and his attitude towards risk ρ_n, but is independent of the aggregate consumption C. For simplicity I assume that both agents have the same time preferences of $\beta_1 = \beta_2 = \beta^* = 0.99$. Thus, the representative agent's utility function has to fulfill (13) for every possible consumption level in θ. I determine the RRA-measure pointwise for each discrete state, giving $u^{*'}(C_\theta) = C_\theta^{-\rho^*} \Rightarrow \left[\frac{C_\theta}{C_0}\right]^{-\rho^*} = \frac{p_\theta}{\pi\beta^*}$ for all $\theta \in \Theta$, so that:

$$\rho^* = \frac{\ln\left[\frac{p_\theta}{\pi\beta^*}\right]}{\ln\left(\frac{C_0}{C_\theta}\right)} \qquad \forall \quad \theta \in \Theta. \tag{14}$$

Note that ρ^* depends on C, because the utility function of the representative agent does not show a constant relative risk aversion coefficient.

In (14) the price p_θ and the amount of aggregate consumption C_θ affect the risk aversion measure ρ^* in the same direction. A growing price leads to a growing ρ^* and a growing aggregate consumption leads to a growing ρ^*. In this case, the problem is that the price p_θ and the consumption C_θ are inversely related. As seen before, a small consumption possibility leads to a higher price.

4.1 Numerical simulations

To determine the resulting effect of a change in the aggregate consumption in one state or of a change in the expected consumption on the representative risk aversion measure ρ^*, I again use numerical simulations.

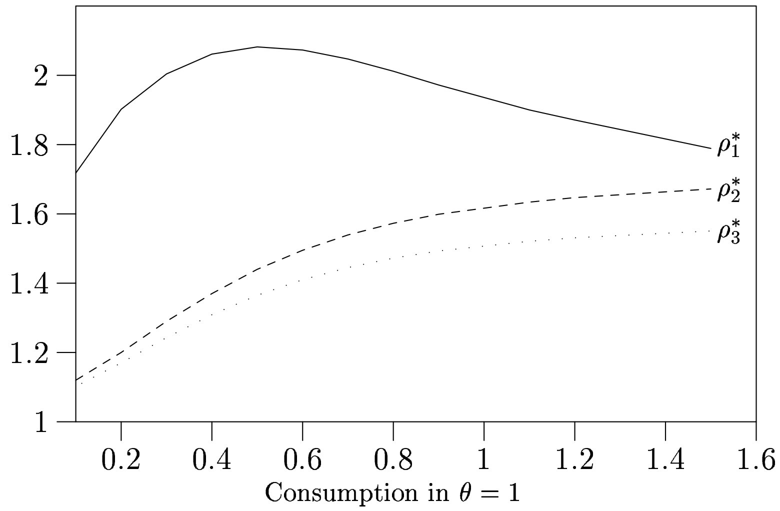

Figure 5: Representative risk aversion ρ^*

In the case of three states of nature I determine ρ^* pointwise for each consumption possibility in the three different states of nature in relation to the output in the first period.

Variation in one state of nature.
In the first situation figure 5 shows the relation between the aggregate payoff in $\theta = 1$ and the corresponding relative risk aversion coefficients ρ^*_θ depending on the state of nature. The solid line depicts the risk aversion measure $\rho^*_{\theta=1}$ with a changing aggregate consumption in this state. $\rho^*_{\theta=1}$ first increases and then decreases with growing consumption. The risk aversion measures corresponding to the other states (the dashed line denotes the risk aversion in the second state and the dotted line that in the third state) increase monotonously with the consumption possibilities.

4.2 The representative risk aversion

Here I show one of the numerical findings of the former section, namely that ρ^* does not approach the risk aversion measure of the more risk averse agent.

Proposition : *If there are two risk averse agents with $\rho_1 < \rho_2$, and there is a small number of discrete future states, then, if $C_\theta \to 0$, the representative agent's risk aversion measure ρ^* does not approach ρ_2.*

Proof: $\theta = j$ is the state of nature in which the amount of consumption is varied. The price p_j has to fulfill the following condition calculated from

(4):

$$\beta_i \pi_j \left[\frac{C_j - c_{2,j}}{1 - c_{2,0}}\right]^{-\rho_1} = \beta_i \pi_j \left[\frac{c_{2,j}}{c_{2,0}}\right]^{-\rho_2} = p_j. \tag{15}$$

If ρ^* approaches ρ_2 , under $\lim_{C_j \to 0}$, we have:

$$\lim_{C_j \to 0} \beta_i \pi_j \left[\frac{c_{2,j}}{c_{2,0}}\right]^{-\rho_2} = \beta^* \pi_j \left[\frac{C_j}{C_0}\right]^{-\rho_2} = p_j \tag{16}$$

Equation (16) could not hold, because from relation (7) between the consumption in t=0 and the initial endowment I find in section (2.1) that the share of consumption of the more risik averse agent is smaller than the one of the less risk avers one with high price, and from relation (10) between the consumption in one state j and the initial endowment in section (2.2) that the share of consumption of the more risik averse agent is higher than the one of the less risk avers one with high price. Thus, the relation of consumption in the first period and the in the state j of the more risk avers agent could not be the same as the relation of the corresponding aggregate consumption with high price. A high price means greather than the aggregate consumption in $t = 0$.

By 14, if $C_j \to 0$, then

$$\lim_{C_j \to 0} \rho^* = \rho_2 = \frac{\ln\left(\frac{p_j}{\beta^* \pi_j}\right)}{\ln\left(\frac{C_0}{C_j}\right)}, \tag{17}$$

or, rearranged

$$-\rho_2(\ln C_0 + \ln C_j) = -\ln p_\theta + \ln(\beta^* \pi_j). \tag{18}$$

Because $C_0 = 1$ equal $\ln C_0 = 0$, we get:

$$p_j = \frac{1}{C_j^{\rho_2}} \beta^* \pi_j \qquad \Leftrightarrow \qquad C_j = \sqrt[\rho_2]{\frac{1}{p_j} \beta^* \pi_j} \tag{19}$$

According to this relation, the price p_j has to grow extremely fast to fulfill the assumption $\rho^* \to \rho_2$ and is $p_j > 1$, if consumption approaches zero and the probanility of occurrence of the state π_j is not arbitrarily small.

So (23) could not be fulfilled and ρ^* could not approach ρ_2. $\square$

A second argument for proposition 2 is the following: From (3) we obtain the two budget restrictions for both agents:

$$c_{1,0} + \sum_{\theta\in\Theta} p_\theta c_{1,\theta} = \alpha_1 \left[C_o + \sum_{\theta\in\Theta} p_\theta C_\theta \right] \quad \forall \quad i = 1,2. \tag{20}$$

From (20), we see that, if $C_j \to 0$, then $c_{1,j} \to 0$, and if $c_{2,j} \to 0$, then $p_j \to \infty$. From the equilibrium condition (15), we see that $c_{1,j}$ has to approach zero faster than $c_{2,j}$.

If the price of consumption p_j is too high in relation to the reciprocal value of consumption, the budget restriction is violated and ρ^* does not approach ρ_2. This occurs if:

$$C_j > \frac{1}{p_j} \tag{21}$$

or

$$\sqrt[\rho_2]{\frac{1}{p_j}\beta^*\pi_j} > \frac{1}{p_j}. \tag{22}$$

Solving for p_j, we and get

$$p_j^{\rho_2 - 1} > \frac{1}{\beta\pi_j} \tag{23}$$

In the case of a small number of states of nature, $\frac{1}{\beta\pi_j}$ is relative small, so that (23) is fulfilled and ρ^* could not approach ρ_2. $\Box$

From (23) we see further that, if the number of states increases ($\#\Theta \uparrow$) so that the probability decreases ($\pi \downarrow$), then ρ^* gets closer to ρ_2 if $C_j \to 0$. In the extreme, if $C_j \to 0$ and if $\#\Theta \to \infty$ then $\pi \to 0$ and $\rho^* \to \rho_2$.[7]

The more risk averse agent cannot obtain sufficient consumption in the critical state of nature to determine the price for the consumption through his risk aversion measure. Thus, even if the consumption possibility approaches zero, the representative risk aversion that generates the same price does not approach the risk aversion coefficient of the more risk averse agent under the assumption of a small number of states.

A numerical example

To see this better, I analyze the case of the previous concrete numerical example of the variation of consumption in the first state. It is assumed that the distribution of aggregate consumption in the second period is $C_1 = 0.1$,

[7] Thus, under the assumption in Benninga/Mayshar (2000), there are sufficiently dense states as a consequence thereof $\pi \to 0$ we obtain the same results as the boundary value.

$C_2 = 2$ and, $C_3 = 3$ units of the single good. According to the simulations, we get a representative risk aversion measure of $\rho^* = 1.723$ for the valuation of the consumption in the first state of nature. In this case, the price for the consumption of one unit of the good is 17.44.

According to the simulations (see figure 5) the representative risk aversion measure ρ^* reaches its maximum, $\rho^* = 2.082$, with a variation of the consumption in $\theta = 1$ at a value of roughly $C_1 = 0.5$. If we assume now that this representative risk aversion measure $\rho^* = 2.082$ remains unchanged with lower consumption possibilities in $\theta = 1$[8], we find the following: A representative risk aversion measure of $\rho^* = 2.082$ with aggregate consumption of $C_1 = 0.1$ leads to a price for this consumption of: $(1/0.1)^{2.082} = 120.226/3 \cdot 0.99 = 39.674 = p_1$. With this price p_1 (the other state prices are $p_2 = 0.1522$ and $p_3 = 0.09867$ [9]), the two agents are initially endowed with $0.5 + 0.05 \cdot 39.674 + 0.1522 + 1.5 \cdot 0.09867 = 2.7839$. If the more risk averse agent keeps his initial endowment of the contingent consumption in $\theta = 1$, he only has 0.8 units of the good to trade in the first period. With this budget she is able to acquire about 0.02 units of contingent consumption in $\theta = 1$ under the price $p_1 = 39.674$ and would own a contingent consumption of $c_{2,1} = 0.02 + 0.05 = 0.07$. In this case he would abstain completely from consumption in the first period and other states. This would not be consistent with his preferences, because he is the more risk averse and wants to smooth his consumption over time and states. From (15) it follows that, with $p_\theta = 39.674$, the more risk averse investor divides his consumption between the first period and the first state at the ratio of $c_{2,0} = 2c_{2,1}$:

$$\left[\frac{C_0}{C_1}\right] = \sqrt[7]{\frac{3p_1}{0.99}} \approx 2. \tag{24}$$

Analogously, the ratios of the other states would be determined. Under $p_2 = 0.1522$ and $p_3 = 0.09867$ the ratios are about $c_{2,0} = 0.8954c_{2,2}$, $c_{2,0} = 0.842c_{2,3}$. According to these ratios, the allocation of consumption for agent 2 in the three states would be $c_{2,0} = 0.14$, $c_{2,1} = 0.0655$, $c_{2,2} = 0.15$, and $c_{2,3} = 0.156$, if the maximum consumption is $c_{2,1} = 0.07$. This is less in every state of nature (and additionally not affordable by the agent) than with the price $p_1 = 17.44$ generated by a representative agent having $\rho^* = 1.723$. In this case the consumption of agent 2 is higher in all states: $c_{2,0} = 0.1597$, $c_{2,1} = 0.0841$, $c_{2,2} = 0.1783$ and $c_{2,3} = 0.1897$.

The more risk averse agent, $\rho_2 = 7$, tries to obtain more consumption in

[8]This is a basic assumption to fulfill the postulate that the representative risk aversion measure ρ^* converges the risk aversion measure of the more risk averse agent ρ_2 as $C_1 \to 0$.

[9]As shown in picture 4 the state prices of the uncritical states do not alter a lot.

the unpleasant state of nature. The high price for this consumption quickly exceeds his budget. None of the investors is able to buy more than a little bit of the critical contingent consumption that they have as an initial endowment because in relation to the price of this critical consumption, the value of their budget left is very small. The prices of consumption in the first period and in the other states of nature are very low in relation to the price of the low consumption in the first state.

Under a lower price, the more risk averse agent achieves less smoothing of consumption over the states, but there is more consumption available in all states. A growing price for the consumption in the first state leads to less consumption in every state for the more risk averse agent. This results from the fact that the consumption in the first period and in the non-critical states of nature is extremely cheap in relation to the consumption in the critical state. But, the more risk averse agent desires to consume in every state, thus he will release the contingent consumption of the first state.

The representative risk aversion measure $\rho^* = 2.082$ occurs with consumption $C_1 = 0.5$. The results of BENNINGA/ MAYSHAR (2000) applied to our model would indicate that the representative risk aversion measure ρ^* grows above this value when $C_1 \to 0$. The described effect would be strengthened under a growing risk aversion. A higher ρ^* leads to a higher price for the critical consumption. Thus, for the investor it becomes more and more difficult to finance consumption in the first state, which exceeds the initial endowment.

5 Conclusion

This paper studies the implications of a change in the consumption possibilities in one state of nature or in the expected consumption in future states on the equilibrium prices and quantities under the regime of heterogeneous agents. If the consumption in one state is very low, the more risk averse agent tries to relocate more consumption to this state. This is done by reduction of the consumption in the first period and in the other states.[10]

The main point of the paper is that, if the heterogeneous agents are replaced by a representative agent without changing the equilibrium prices, the representative agent displays first an increasing and then a decreasing relative risk aversion under changing aggregate consumption.

If the consumption possibility gets lower, the more risk averse agent does not insist on his preferences for a smoothed consumption. It is better for him to achieve more consumption in every state, though more unequally

[10] This effect was neglected in the paper of BENNINGA/ MAYSHAR (2000).

distributed. He acts as if he were "quasi more risk loving". The less risk averse agent is not able to influence this behavior. He is not so much interested in the consumption in the critical state and so does not force the corresponding state price to climb. In fact, a higher price would be better for the less risk averse agent, because he would have more consumption in every state of nature[11]. This situation would be preferred by him.

Thus, the representative agent displays hump shaped preferences. This fact could explain the inconsistence of the Black-Scholes formula with empirical prices as done in BENNINGA/ MAYSHAR(2000) or FRANKE/ STAPLETON /SUBRAHMANYAM (1998), presuming decreasing relative risk aversion of the representative agent.

Extensions of the model, such as allowing for more states of nature or more periods, are left for future research.

A Share of consumption in t=1

From the restriction (3), by normalizing $C_0 = 1$ and $c_{i,0} = c_{i,0}/C_0$:

$$c_{i,0} + \sum_{\theta=1}^{\theta_n} p_\theta c_{i,\theta} = \alpha_i \left[1 + \sum_{\theta=1}^{\theta_n} p_\theta C_\theta \right]. \tag{25}$$

Rearranging (4), we get

$$c_{i,\theta}^{\rho_i} = \left[\frac{\beta_i \pi_\theta}{p_\theta} \right] c_{i,0}^{\rho_i} \quad \Leftrightarrow \quad c_{i,\theta} = \left[\frac{\beta_i \pi_\theta}{p_\theta} \right]^{\frac{1}{\rho_i}} c_{i,0}.$$

Thus, from (25), we have

$$c_{i,0} + \sum_{\theta=1}^{\theta_n} p_\theta \left[\frac{\beta_i \pi_\theta}{p_\theta} \right]^{\frac{1}{\rho_n}} \alpha_i = \alpha_i \left[1 + \sum_{\theta=1}^{\theta_n} p_\theta C_\theta \right].$$

By solving for α_i we obtain the relation (7):

$$\alpha_i = c_{i,0} \left[\frac{1 + \sum_{\theta=1}^{\theta_n} p_\theta [\beta_i \pi_{i\theta} / p_\theta]^{1/\rho_i}}{1 + \sum_{\theta=1}^{\theta_n} p_\theta C_\theta} \right]. \tag{26}$$

[11] A small share of aggregate consumption for one agent means in our two investor model a bigger share for the other agent, $c_{1,\theta} + c_{2,\theta} = C_\theta$.

B Share of consumption in t=2

For an arbitrary state of nature $\theta = j$, $j \in \Theta$, the share of consumption could be calculated. Market clearing for the contingent consumption in $\theta = j$ gives $\sum_{i=1}^{I} c_{i,j} = C_j$. From (4) solved for $c_{i,\theta}$, we have

$$c_{i,\theta} = c_{i,0} \left[\frac{\beta^* \pi_\theta}{p_\theta} \right]^{\frac{1}{\rho_i}}, \tag{27}$$

and for $c_{i,0}$:

$$c_{i,0} = c_{i,\theta} \left[\frac{p_\theta}{\beta^* \pi_\theta} \right]^{\frac{1}{\rho_i}}. \tag{28}$$

From the budget restriction, we have

$$c_{1,0} + \sum_{\theta \in \Theta} p_\theta c_{1,\theta} = \alpha_1 \left[C_o + \sum_{\theta \in \Theta} p_\theta C_\theta \right]. \tag{29}$$

Replacing $c_{1,0}$ by (28) and each $c_{i,\theta}$ by (27) in equation (29) and considering $\pi_1 = ... = \pi_{\theta_n}$, results in

$$c_{i,j} \left[\frac{p_j}{\beta^* \pi} \right]^{\frac{1}{\rho_i}} + p_j c_j + \sum_{\theta \in \Theta \backslash j} p_\theta c_{i,j} \left[\frac{p_j}{\beta^* \pi} \right]^{\frac{1}{\rho_i}} c_{i,0} \left[\frac{\beta^* \pi}{p_\theta} \right]^{\frac{1}{\rho_i}}$$
$$= \alpha_1 \left[C_o + \sum_{\theta \in \Theta} p_\theta C_\theta \right]. \tag{30}$$

Simplifing (30) gives

$$c_{i,j} \left[\left[\frac{p_j}{\beta^* \pi} \right]^{\frac{1}{\rho_i}} + p_j + \sum_{\theta \in \Theta \backslash j} p_\theta \left[\frac{p_j}{p_\theta} \right]^{\frac{1}{\rho_i}} \right] = \alpha_1 \left[C_o + \sum_{\theta \in \Theta} p_\theta C_\theta \right]. \tag{31}$$

Rearranging (31) leads to the desired share of consumption in state j depending on the initial endowment:

$$c_{i,j} = \alpha_i \left[\frac{1 + \sum_{\theta \in \Theta} p_\theta C_\theta}{p_j + \left[\frac{p_j}{\beta^* \pi} \right]^{\frac{1}{\rho_i}} + \sum_{\theta \in \Theta \backslash j} \left[\frac{p_j}{p_\theta} \right]^{\frac{1}{\rho_i}}} \right] \tag{32}$$

Analogously, the shares of consumption in the other states of nature are determined.

References

[1] Simon Benninga and Joram Mayshar. Heterogeneity and option pricing. *Review of Derivatives Research*, 4:7–27, 2000.

[2] George M. Constantinides. Intertemporal asset pricing with heterogeneous consumers and without demand aggregation. *Journal of Business*, 55(2):253–267, 1982.

[3] Gerard Debreu. Existence of competitive equilibrium. In K. J. Arrow and M. D. Intriligator, editors, *Handbook of Mathematical Economics*, volume II, chapter 15, pages 697–743. North-Holland, Amsterdam, 1982.

[4] Jerome Detemple and Shashidar Murthy. Intertemporal asset pricing with heterogeneous beliefs. *Journal of Economic Theory*, 62:294–320, 1994.

[5] Bernard Dumas. Two person dynamic equilibrium in the capital market. *The Review of Financial Studies*, 2(2):157–188, 1989.

[6] Günter Franke, Richard C. Stapleton, and Marti G. Subrahmanyam. Who buys and who sells options: The role of options in an economy with background risk. *Journal of Economic Theory*, 82:89–109, 1998.

[7] Jerome Detemple and Larry Selden. A general equilibrium analysis of option and stock market interactions. *International Economic Review*, 32:279–303, 1991.

[8] Kenneth L. Judd. *Numerical Methods in Economics*. MIT Press, Cambridge, 1998.

[9] Alan P. Kirman. Whom or what does the representative individual represent? *Journal of Economic Perspectives*, 6(2):117–136, 1992.

[10] Hayne E. Leland. Who should buy portfolio insurance? *Journal of Finance*, 35:581–594, 1980.

[11] Robert E. Lucas. Asset prices in an exchange economy. *Econometrica*, 46:1426–1446, 1978.

[12] Jiang Wang. The term structure of interest rates in a pure exchange economy with heterogeneous investors. *Journal of Financial Economics*, 41:75–110, 1996.