Divergence of credit valuation in Germany – Continuous theory and discrete practice –

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Abstract

Lending is associated with credit risk. Modelling the loss stochastically, the cost of credit risk is the expected loss. In credit business the probability that the debtor will default in payments within one year, often is the only reliable quantitative parameter. Modelling the time to default as continuous variable corresponds to an exponential distribution. We calculate the expected loss of a trade with several cash flows, even if the distribution is not exponential. Continuous rating migration data show that the exponential distribution is not adequate in general. The distribution can be calibrated using rating migrations without a parametric model. A practitioner, however, will model time as a discrete variable. We show that the expected loss in the discrete model is a linear approximation of the expected loss in the continuous model and discuss the consequences. Finally, as costs for the expected loss cannot be charged up-front, the credit spread over risk-free interest is derived.

1 Introduction

Credit risk attracts immense interest in finance at present. The key reason is that insolvency rates have increased in the last years. E.g. the annual rate for the commercial sector in Germany has increased from 0.1% in the 1960s to more than 1% in the first years of the twenty first century (**??**). As a consequence, national authorities, such as the "Bundesaufsichtsamt für das Kreditwesen" (BaKred) in Germany, and international authorities, such as the "Bank of International Settlement" in Basle, strive to regulate the management and measurement of credit risk tighter[3]. A first step in that

[3] The development is similar to the regulation of management and measurement of market risk. For the latter the international "minimal requirements for trading" have evolved in the past ten years and have been integrated in German law ("Mindestanforderungen für das Betreiben von Handelgeschäften" (MaH)).

direction on the international level is the new capital accord of the Bank of International Settlement (named Basle II, see ?).

Besides the regulatory efforts, the market simultaneously has developed financial products which allow the trade of credit risk. The volume of these products has rapidly grown because of the increase in demand to secure credit risk. A standardization in documentation, e.g. for credit default swaps (see ? and ?) has enabled such growth. The leverage of credit risk can be dramatically higher for those credit derivative products compared with genuine lending. Investment banks manage these high risks because of regulatory requirements, but also voluntarily, with sophisticated methods of primarily stochastic and statistical nature, see e.g. ?. As a consequence, the lending business is now in competition with investment banking and needs to adopt the developed methodologies or at least be aware of the inaccuracy of simplistic methods. The present paper tries to shed light on these inaccuracies by postulating that the sophisticated methodology is a continuous-time markov model and the simple approach is based on discrete univariate statistics.

In fact, some simplifications are indeed correct in the lending business. E.g. it is typical in practice to fix the payment dates and payment heights when the contract is negotiated. Hence, we refrain from modelling random payment dates and heights. Additionally, a loan starts with the granting of the notional, say today, and ends with maturity.

The credit risk, i.e. the loss that is associated with the insolvency of the debtor costs the creditor money. When being contacted by a potential debtor at first time a creditor needs to calculate the cost in order to decide whether, and if at which rate, to lend. He will base the decision usually on data of similar cases he has been able to study. The loss will depend on the time of default after the initiation of the loan. The analysis of time-to-event data is well established under a variety of labels, e.g. "event history analysis", "survival analysis" or "analysis of failure times" (see e.g. ?, ?, ?, ?, ?, ? and ?).

Our contribution is now the adaption of those methods: We derive accurate formulae for the expected loss, constituting the primary stochastic cost when lending. To this end, we show that the loss arising from a loan is a stochastic (jump) process. We compare the costs with the costs arising from discrete modelling theoretically and in an example. Finally we propose non-parametric estimation for the calibration of the costing formulae and illustrate that in an example using rating data.

Three different situations are conceivable. The hazard of default can be either constant over the whole period or it can be modelled as deterministic function changing over time. Furthermore, the payment of the debtor can either be a single payment, a so called bullet, at the time of maturity or it can more realistically be modelled as incorporating many payments at fixed a priori known time points till maturity.

The paper is structured as follows. In the next section we consider the most simple though unrealistic case of a constant hazard rate as well as a single payment scheme. This is mathematically spoken the most simple case and is therefore well suited to introduce the problem and the basic ideas. In section three we generalize this setup by first considering a payment scheme with many payments till maturity by assuming the hazard rate to be constant over time. Afterwards we generalize this model by furthermore allowing the hazard rate to be a non-constant deterministic function. The case of a bullet payment and a non-constant hazard rate is not relevant in practise and does not give any additional theoretical insight and is therefore omitted here.

In practice the costs of a loan are usually obtained by applying a discrete univariate model. Section four compares our results with this discrete model. Section five finally provides a test for the homogeneity of the loss process. Furthermore, kernel estimation of the default hazard is proposed if inhomogeneity must be assumed and is exemplified using rating data. Section six concludes, proofs are given in the appendix.

2 A simple model

We take the point of view of a creditor who lends a certain amount, denoted by "notional". Our prototype loan that will serve to illustrate the problem is particulary simple. It originates in $t_0 = 0$ and ends at maturity T. The notional is 1 and payed (back) in T. The creditor is prepared to reduce the initial payment by the expected loss considered as cost. Our aim is to calculate the expected loss. The credit loss from the perspective of the creditor for $0 \leq t \leq T$ is given by the stochastic process

$$L_t := I_{\{\tau \leq t\}},$$

where $I_{condition}$ denotes the indicator function, i.e. is 1 if the *condition* is true and 0 otherwise. The default time of the debtor is denoted by τ. We require a probability space $(\Omega, P, \mathcal{F})$, an increasing filtration of sub-σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ and L_t to be adapted to the filtration. In this section we will consider the particulary simple case of $\tau \sim Exp(\alpha)$, i.e. assume the loss process to be homogeneous. The generalization to an arbitrary deterministic default hazard $\alpha(t)$ will be part of the next section.

It is well known that any stochastic process can be decomposed into a trend-type component, named "compensator", and a martingale. In order to determine the compensator, we use that

$$E(dL_t|\mathcal{F}_{t-}) = I_{\{\tau \geq t\}} P(\tau \in [t, t+dt] | \tau \geq t) = I_{\{\tau \geq t\}} \alpha dt$$

with intensity process $\lambda(t) := I_{\{\tau \geq t\}}\alpha$ and left-side limit $\mathcal{F}_{t-} := \lim_{dt \searrow 0} \mathcal{F}_{t-dt}$.

Hence, we have $L_t = \Lambda_t + M_t$ with $\Lambda_t = \int_0^t \lambda(s)ds$ (see **?**, pg. 51) as cumulative intensity, i.e. the compensator. Clearly, $L_0 = 0$ almost surely and $\Lambda_0 = 0$ almost surely so that $E(M_T) = E(M_0) = 0$. Denoting the density of the default time τ by $f(t)$ we have

$$\begin{aligned} E(L_T) &= E(\Lambda_T) + E(M_T) = \int_0^T EI_{\{\tau \geq s\}} \alpha ds = \int_0^T f(s)ds = F(T) \\ &= 1 - e^{-\alpha T}. \end{aligned} \tag{1}$$

The notional of 1 was chosen for pure convenience. For a generalization to any (known) cash flow A, multiply the expected loss (1) by A. So far, our understanding has been that the counterparts, creditor and debtor, agree to exchange the amount A, the notional, at maturity T. Consequently, the creditor reduces the initial payment to the debtor by the cost for the expected loss and hence pays out $Ae^{-\alpha T}$. However, to regain the notional at the maturity of a trade is common only in the bond market. For loans, the notional is usually paid to the debtor at origination and returned including interest at maturity. The latter modality is still a simplification. Usually, only short running loans have an aggregated compensation at maturity called “bullet” structure. For longer maturities it is common to pay interest periodically. We will cover this case in the next section. For the application of (1) to a loan with final compensation think of the final payment A as the compensation for a loan with notional $Ae^{-\alpha T}$. If now, instead of the final payment A, the pay-out at origination B is fixed, A (seen as redemption including costs for the expected loss) is

$$A = Be^{\alpha T}. \tag{2}$$

Formula (2) demonstrates that the accounting for the expected loss is similar to continuous compounding of A with interest rate α (see **?**, pg. 46). The latter is known in the pricing of derivative products subject to credit risk (**?**, **?**) or (see **?**, pg. 106).

3 The multiple payment loan

The last section displayed the general idea. Here we want to enlarge the model to representative situations. Typically, loans involve periodical payments, it can be viewed as a cash flow series a_{t_i} of payments at the times $0 = t_0 < t_1 < \ldots < t_n = T$. Positive payments symbolize flows from the debtor to the creditor, negative payment from the creditor to the debtor. Figure 1 displays a typical contract.

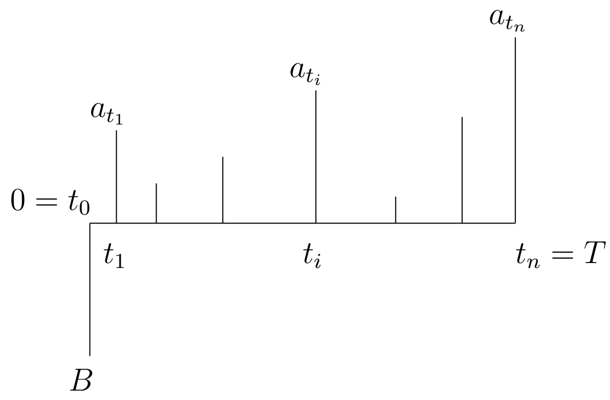

Figure 1: Typical cash flow series of a loan

The loss is now the stochastic process

$$L_t = I_{\{t \geq \tau\}} \sum_{t_i \geq \tau} a_{t_i} = I_{\{t \geq \tau\}} \sum_{i=1}^{n} I_{\{\tau \leq t_i\}} a_{t_i}.$$

In banking the creditworthiness of a debtor is usually described by a rating defining the one-year probability of default (PD_1). E.g. rating agencies assess these probabilities and publish their results for further uses by credit originators (see e.g. **???**). As a result, PD_1 is the most well-known parameter in credit risk and we like to display the expected loss in terms of the PD_1. As the default hazard is constant $PD_1 = P(\tau \leq 1)$ is determined by α is given by

$$F(1) = 1 - e^{-\alpha}, \tag{3}$$

(see e.g. **?**) where $F(t) := P(\tau \leq t)$. Any PD of the debtor is given by $F(t) = 1 - e^{-\alpha t}$.

Theorem 1 *Let the series $a_{t_0}, \ldots, a_{t_n}$ represent the payments of a loan contract originated in t_0 maturing in t_n. Assume that the default risk of the* *debtor is constant and given by the one-year probability of default PD_1. Than the expected loss of the creditor a priori, i.e. at t_0 is*

$$E(L_T) = \sum_{i=1}^{n} (1 - (1 - PD_1)^{t_i}) a_{t_i}.$$

The proof is postponed to the appendix.

In the remainder of the section we will account for two modifications, the time-value of money and the variability of the default hazard. The principle of time-value in money is that future payments are less valuable now, in t_0. We must discount the payments with the so-called risk-free interest rate. We must discount the contractual payments a_{t_i} with a risk-free interest rate curve $r(t)$. By using continuous compounding we get

$$a_{t_i} e^{r(t_i)t_i}. \tag{4}$$

In fact, we know the interest rate curve $r(t)$ for lending free of credit risk in t_0. One possibility is to use the interbank offer rates, e.g. the "Euribor" for the Euro-denoted loans. Hence, the discounted payments are again deterministic and do not need methodological advances.

The second generalization is to allow for a non-constant default hazard. Often in practice it is argued that the one-year probability of default PD_1 cannot be applied to all time periods of the loan as done so far. In terms of the default hazard the assumption of a constant rate α is unrealistic, we need a time-dependent default hazard $\alpha(t)$.

Theorem 2 *Let the series $a_{t_0}, \ldots, a_{t_n}$ represent the payments of a loan contract originated in t_0 maturing in t_n. Assume that the default risk of the debtor is given by the known default hazard $\alpha(t)$. Let $r(t)$ denote the continuous compounding interest rate at time t_0 for loan free of credit risk. Than the expected loss of the creditor at t_0 is*

$$E(L_T) = \sum_{i=1}^{n} (1 - e^{-\int_0^{t_i} \alpha(s)ds}) a_{t_i} e^{r(t_i)t_i}.$$

Proof: It should be noted that

$$E(dI_{\{\tau\leq t\}}|\mathcal{F}_{t-}) = I_{\{\tau\geq t\}}P(\tau \in [t, t+dt]|\tau \geq t) = I_{\{\tau\geq t\}}\alpha(t)dt$$

with intensity process $\lambda(t) := I_{\{\tau\geq t\}}\alpha(t)$. Similar to the proof of Theorem 1 we have $E(L_T) = I_{\{\tau\geq t\}} \sum_{t_i\geq t} a_{t_i} e^{r(t_i)t_i}\alpha(t)dt$. Again, we have achieved $E(L_T) = \sum_{i=1}^{n} F(t_i)a_{t_i}e^{r(t_i)t_i}$ where the cumulative distribution function $F(t)$ is no longer the exponential distribution but one-to-one determined by the default hazard via $F(t) = 1 - e^{\int_0^t \alpha(s)ds}$. □

4 Comparison to the discrete model

Credit Risk is in practice often modelled in a discrete fashion (see e.g. **?**), e.g. with one year intervals. Consider now the prototype loan type of the previous section that has a final compensation of 1 at maturity T. Market standard is now to assume equivalence of a loan over T years to T independent one-year loans and calculate the expected loss as[4]: (using 3)

$$T \times F(1) = T(1 - e^{-\alpha}). \tag{5}$$

The stated procedure is underpinned statistically by the Bernoulli model for the one-year default and an implicit assumption is that the Bernoulli probability does not change, i.e. that default hazard is again constantly α. To compare the exact expected loss costs (1) with the market method (5) consider the difference

$$D_T := T(1 - e^{-\alpha}) + e^{-\alpha T} - 1.$$

It can be easily seen that the roots of D_T, i.e. the maturities where the market method is exact, are $T = 0$ and $T = 1$. To examine the behavior of

[4] Hierfür wäre natürlich (neben meiner Gewissheit aus der WestLB) auch noch ein Zitat nützlich.

D_T in more detail we consider the first derivative

$$D'_T = (1 - e^{-\alpha}) - \alpha e^{-\alpha T}$$

and obtain the local extremes by solving $D'_T = 0$. This is the case for $T_0 = -\frac{1}{\alpha} ln(\frac{1-e^{-\alpha}}{\alpha})$. The second derivative of D_T is $D''_T = \alpha^2 e^{-\alpha T}$ being positive at $T = T_0$ proving that the difference D_T has exactly one local minimum. This proves that D_T is decreasing for $T < T_0$ and increasing for $T > T_0$ showing that D_T is negative for $0 < T < 1$ and positive for $T > 1$. Therefore, the market method understates the costs for maturities under one year and overstates the costs for maturities over one year where the benefits from our approach increases exponentially. Numerical root finding reveals that the cost of the market method (5) values 10% higher compared with the exact cost (1) when the PD_1 is 1% and the maturity $T = 20$.

Given the fact that T_0 is near to zero for all $0 < \alpha < 1$, which holds for the realistic cases of $0 < PD_1 < 0.2$, our approach is analogous to the one-year approach for $T < 1$ (year).

The market method of expected-loss calculation (5) for our prototype loan is easily generalized to the multi-payment loans of Theorem 1. One must simply consider each payment a_{t_i} to represent a prototype loan and assume all prototype loans to be independent. In line use the one-year probability of default PD_1, apply it to each year in which a loan is granted and add the yearly costs $\text{PD}_1 \times$ notional. Interestingly, the procedure is an analytic approximation of the exact cost given in Theorem 1.

Theorem 3 *Under the assumptions and notations of Theorem 1 holds:*

$$EL_T \approx \sum_{i=1}^{n} PD_1 t_i a_{t_i}.$$

The equality holds for $PD_1 = 0$.

Proof: Expand the coefficient $1 - (1 - PD_1)^{t_i}$ in Theorem 1 with respect

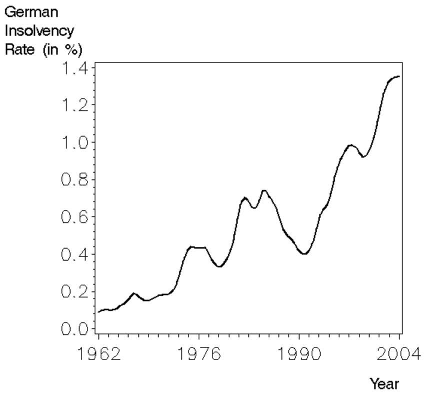

Figure 2: Annual insolvency rate for the German economy from 1962 to 2004 (until 1994 West Germany) in per thousand.

to the variable PD_1 at 0:

$$1-(1-PD_1)^{t_i}=\sum_{\nu=1}^{\infty}(-1)^{\nu+1}\frac{t_i!}{(t_i-\nu)!}PD_1^{\nu}.$$

If we use a linear approximation, namely $\nu=1$, we have

$$1-(1-PD_1)^{t_i}\approx t_iPD_1.$$

□

Thus, we have shown that the conventional calculation is a Taylor approximation, only using the linear coefficient.

Clearly, the approximation is good for small default probabilities. However, insolvency rates, and hence default probabilities, has been rising in Germany for the past forty years. Figure 2 displays the annual insolvency rate for companies (**??** amended by the recently published value for 2004).

Whereas in the sixties and seventies the approximation given in Theorem 3 may have been sufficiently accurate, nowadays the approximation seems to be questionable. The average insolvency rate in the sixties of the last century was only around 10% of the average rate in the first decade of the present century.

We would like to investigate potential business implications. Our example is a 10 year loan of 100 million Euro notional to a counterpart with one-year PD of 1%. We consider the case of constant annuities, i.e. annual payments of 10 million to pay back the debt. The expected loss with respect to the market method (Theorem 3) is 5.5 million Euro, or 5.5%. Whereas the exact expected loss in the exact calculus (Theorem 1) is 5.34 million Euro - or 5.34%. The user of the market method collects $160,000$ Euro too much from the counterpart. The difference of 16 basis points is huge from an investment banking perspective, especially when looking at liquid markets. The loan market is already liquid and becomes more and more liquid with current increases in loan trading. Even more liquid is the interest rate derivative market, especially the Swap market.

However, the quantity is not easy to assess because it is not denoted on an annual basis. The interest rate needs distribution over ten years. It is not possible to divide the amount by ten in the case of the ten-year loan because the future cash flows are subject to default, and, hence less valuable. The equilibrium premium mark-up is a constant surplus ε to the a_{t_i}. In the continuous model, the additional cash flow arising from the risk premium is $P_t = \varepsilon \sum_{i=1}^{n} 1_{\{\tau > t_i\}}$ and in order to compensate for the expected loss in Theorem 1, the expected premium needs to equal the expected loss. The expected premium $E(P_T)$ is easily calculated for a constant default hazard as $\varepsilon \sum_{i=1}^{n} e^{-t_i \alpha}$. The premium that the credit needs to collect, the "credit spread", is[5]

$$\varepsilon = \frac{\sum_{i=1}^{n}(1-(1-PD_1)^{t_i})a_{t_i}}{\sum_{i=1}^{n}(1-PD_1)^{t_i}}. \tag{6}$$

For our example of the ten year loan with a 1% annual default probability, the ε is 0.5639%, which adds up to 5.639 million if the counterpart fulfills all duties. This amount is slightly higher than the expected loss of 5.34 million due to the potential loss of parts of the premium when the counterpart default prior to the maturity of the trade. The easiest charge for the expected loss on an regular basis as indicated by the formula (5) for the one-year loan would be 1% for all periods. This approach would almost double the exact credit spread (6) in our example.

5 Calibrating the default hazard

We see that pricing for credit risk needs a specification of the default hazard, either as in Theorem 1 and 3 of a constant hazard or as in Theorem 2 of a variable hazard. The question arises which hazard function to use. The assumption of a constant rate is certainly appealing in terms of simplicity. Having historical data at hand, the maximum likelihood estimate from a sample $\tau_1, \ldots, \tau_n$ of identically independently distributed default times is known to be $n/\sum_{i=1}^{n}\tau_i$. The point is, rather, how can we decide whether the hazard rate is constant?

The rating class assigns a counterpart to a homogeneous population. Historical behavior of the population can be used to infer about the shape of the hazard function. The first important question is: When is the origin of

[5]From microeconomic theory we know that in competitive markets, as in the capital market, the supplier of capital has negligible impact on the price. However, proper pricing of the loan is not redundant for one important reason. The price developed can be seen as a hurdle rate to the market price. If the credit spread in the market is above the hurdle rate (6) we safely enter into a deal, otherwise our risk perception is above the market's and the investor should refrain from trading.

time? In order to observe default times we must define a point in time when we start counting. Banks must rate a counterpart internally when a (first) loan is negotiated. The interesting event is "default after amendment to the portfolio" the time of initialization of the trade must be seen as the origin of the random variable "default time". Ongoing regular rating at least on an annual basis is mandatory until the trade is matured. Default dates are usually known exactly. These data can be used to test the assumption of a constant default hazard for the transition from the respective rating class to the default state. Fortunately, from a business perspective, many of the engagements end without the observation of a default time. The censoring that can occur is then right-censoring as the event may not be observed due to the termination of trade without default[6]. In any type of survival analysis and especially in assessing the credit risk, right-censored observations arise from "good" risk, meaning that counterparts do not default while being under investigation. These observations have to be incorporated into the estimation and testing of the default hazard because neglecting them would result in serious bias, in this case over-costing. Interestingly, **?** uses the same continuous type of data (to test for Markovian behavior, meaning rating drift), although with calender dates as time axis.

Employing the common notation for censored data, we define that the n independent observations $X_i = \max\{\tau_i, c_i\}$, $i = 1, \ldots, n$, refer either to the default times τ_i or the censoring times c_i and that $\delta_i = I_{\{X_i = \tau_i\}}$ indicates the censoring for $i = 1, \ldots, n$. Ordering of the censoring indicators $\delta_{(i)}$ follows that of the corresponding observations $X_{(i)}$.

We suggest, at first stage, to check graphically for a constant hazard rate, i.e. a linear cumulative hazard function $A(t) := \int_0^t \alpha(s)ds$. A standard plot in statistical software (in the presence of right-censoring) data is $-\log$ of the product-limit estimate. The latter estimates the survival function, meaning 1 minus the cumulative distribution function, without bias (see **?**). See again

[6]Other censoring vents are discontinued rating or migration to another rating class.

equality (3) for the relation. If it appears to be linear, the hazard function might be constant. A kernel-smoothed estimates of the hazard rate itself can easily be implemented as we will see in the example.

However, an even better indicator of the linearity for the hazard function is an inferential test against the null hypothesis of a constant hazard function (given the one-year PD_1).

In the statistical literature the test for a constant hazard function is known as "one-sample log-rank test" (see **?**). The test can be embedded in the context of counting processes (see **?**). As we can assume the one-year PD_1 to be known and, see (3), apply the test to the hypothesis $H_0 : \alpha(t) \equiv -\log(1-PD_1)$. The asymptotically standard normal distributed test statistic is $V = (N(t) - E(t))/\sqrt{E(t)}$ where $N(t) := \sharp\{i : T_i \leq t, C_i = 1\}$ is the number of uncensored defaults until time t and $E(t) := -\log(1 - PD_1)\sum_{i=1}^{n}(X_i \wedge t)$ is the expected number of defaults in $[0, t]$. Usually, one uses for the argument t the largest uncensored default time.

As an example we use internal rating data of a rating system, made available to us by WestLB AG, Düsseldorf. The data set comprises 200 counterparts belonging to a homogeneous (minor) rating class. Default or censoring times range from 7 days to 1789 days (4.9 years). The censoring (withdrawn rating, discontinued business relation, change of rating class other than default, still active) is 69.5%. Figure 3 displays estimates of the survival function and the cumulative hazard function. As in the right tail, above 678 days, no uncensored information is available, resulting in uninformative estimates.

The estimation of the cumulative hazard function looks rather concave than linear. We need to estimate the survival curve for the test, later.

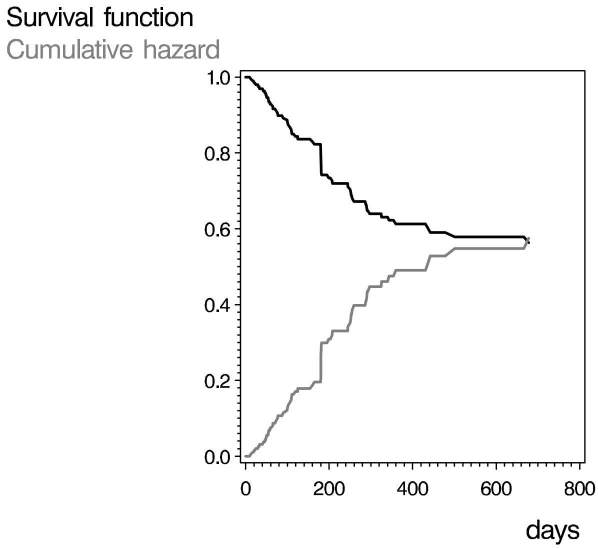

Figure 3. *Kaplan-Meier estimate of the survival distribution function of the default time in the rating class (black line) and corresponding estimate for cumulative hazard function (grey line).*

We estimate the default hazard (used in Theorem 2) via kernel smoothing. We use the following kernel estimate of the hazard rate (see e.g. **?**)

$$\alpha_n(t) = \sum_{i=1}^{n} \frac{\delta_{(i)}}{n-i+1} \frac{1}{R_{NN}(X_{(i)})} K\left(\frac{X_{(i)} - t}{R_{NN}(X_{(i)})}\right). \tag{7}$$

with a bi-quadratic kernel $k(\cdot)$ (see e.g. **?**) and nearest-neighbor bandwidth $R_{NN}(\cdot)$ (see e.g. **?**) to assess the assumption of a constant hazard. For a proof of strong consistency see **?**.

For an optimal choice of the bandwidth parameter, namely the number of nearest neighbors, we use the selector from the density estimation context. The bandwidth was chosen optimally with respect to the mean integrated squared error in the context of density estimation assuming an underlying normal distribution. This method is sometimes referred to as "rule of thumb". (See for example **?** for the case of fixed bandwidth density estimation.) A bridge, laid out in detail in **?**, helps us to calculate the optimal number of nearest neighbors from an optimal fixed bandwidth.

For our study data the hazard rate estimate programmed in SAS/IML is depicted in Figure 4. We restrict the display to 800 days, as in the case of Figure 3. It can be seen that three modes seem to be present.

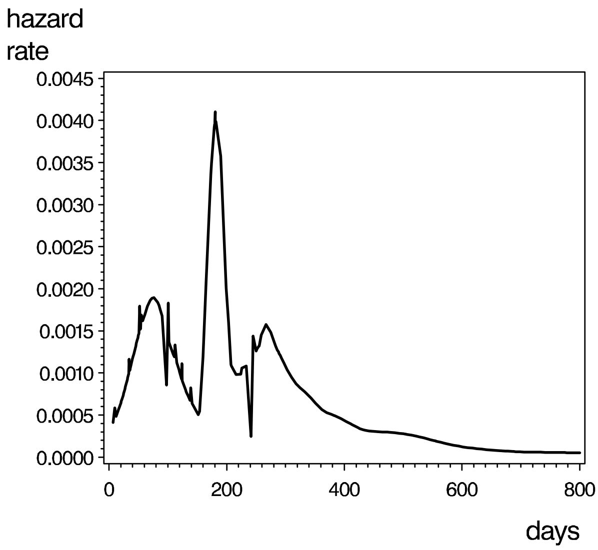

Figure 4. *Kernel estimate of the hazard rate with nearest neighbor bandwidth and* 23 *nearest neighbors.*

To supplement the descriptive assessment of the hazard rate we use the log-rank test for inferential statistics as implemented in SAS/STAT.

We assume the rating to be well calibrated and use the one-year probability of the product-limit estimate of 38.7% as a one-year PD which is assumed to be known (see Figure 3). The maximum uncensored survival time t is 1.86 years and until then $N(t) = 61$ defaults occur. Under the null hypothesis of a constant default hazard the expected number is $E(t) = 77$. The test rejects the hypothesis at a level of 5% and, hence, strongly supports our judgement of a varying hazard rate. The p-value is 0.034.

6 Summary

We have discussed the loss and its costs in terms of the expected loss of loans. We started with a loan with bullet payment and modelled the loss as stochastic jump process to derive a formula for the exact expected loss. Assuming a constant hazard rate of default for the counterpart, we showed that the standard practice in banking is a Taylor approximation of our result only by using the linear component. We gave evidence that the hazard rate may not be assumed to be constant and derived a formula which accounts for the cumulative hazard rate. Furthermore, we showed how to estimate the latter from right-censored data. The techniques which we applied generalize the calculation of the expected loss by using migration matrices as in **?** due to the continuous nature which makes the assumption of a discrete cash flow structure such as for rating matrices redundant. We have integrated the cost of default into the interest rate charged by the investor as price building.

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A Proof of Theorem 1

To calculate the compensator, we calculate again the expected incremental changes

$$\begin{aligned} E(dL_t \mid \mathcal{F}_{t-}) &= I_{\{\tau \geq t\}} \sum_{t_i \geq t} a_{t_i} E(I_{\{\tau \in [t, t+dt]\}} \mid \mathcal{F}_{t-}) \\ &= I_{\{\tau \geq t\}} \sum_{t_i \geq t} a_{t_i} \alpha dt \end{aligned}$$

where $\lambda(t) := 1_{\{\tau \geq t\}} \sum_{t_i \geq t} a_{t_i} \alpha$ is the intensity process. To calculate the expected loss up until time T in the stochastic calculus we need to calculate the expected trend at time T because

$$EL_T = E\Lambda_T.$$

Now,

$$\begin{aligned} E\Lambda_T &= E\int_0^T \sum_{t_i \geq s} a_{t_i} I_{\{\tau \geq s\}} \alpha ds \\ &= \int_0^T \sum_{t_i \geq s} a_{t_i} E I_{\{\tau \geq s\}} \alpha ds \\ &= \sum_{i=1}^{n} a_{t_i} \int_0^T I_{\{t_i \geq s\}} f(s) ds \\ &= \sum_{i=1}^{n} F(t_i) a_{t_i} \\ &= \sum_{i=1}^{n} (1 - e^{-t_i(-\log(1-PD_1))}) a_{t_i}. \end{aligned}$$

□