Anticipated Shocks in Continuous-time Optimization Models: Theoretical Investigation and Numerical Solution

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Abstract

We derive the well-known continuity principle for adjoint variables for preannounced or anticipated changes in parameters for continuous-time, infinite-horizon, perfect foresight optimization models. For easy and intuitive numerical computation of the resulting multi point boundary value problem we suggested to simulate the resulting differential algebraic system representing the first order conditions. By ensuring that the state variables and the adjoint variables are continuous, potential jumps in the control variables are calculated automatically. This can be easily conducted with the relaxation algorithm as proposed by Trimborn et al. (2007). We solve a Ramsey model extended by an elementary Government sector numerically. Simulations of a preannounced increase in the consumption tax show a qualitative different pattern depending on the intertemporal elasticity of substitution.

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1 Introduction

Dynamic macroeconomic theory very often assumes the agents in the model to experience perfect foresight. For example, a representative consumer maximizes discounted utility subject to his or her budget constraint over an infinite horizon. Then, it is assumed that at the point of time when the maximization takes place, the maximizing agent is aware of the whole set of information. For example, the agent knows the future time path of variables that are exogenous to him. Then, following Bellman's principle, the decisions are time-consistent, that is, at a later point of time the same solution is optimal if the set of information did not change.

Changes in the underlying parameters, e.g. tax rates or preference parameters are frequently analyzed by assuming a sudden, immediate shock. More precisely, at a specific point of time, say t_0, new parameter values are applied and the optimizing agent experiences this information at exactly the same point of time. Then, he or she can optimize over the remaining time horizon. Since the point of time of the shock and the point of time of the information propagation coincide, i.e. the shock is not preannounced or anticipated, control variables can jump at t_0. State variables cannot jump at t_0 by construction.[1]

This simplifying assumption is very useful to analyze e.g. policy measures in a stylized way, however, in some cases it may be oversimplifying. The reason is that usually the time of the parameter changes and the information propagation to the agent do not coincide. For example, policy measures are usually announced some time before they take place. Therefore, the stylized analysis of unexpected shocks cannot take account for anticipatory actions conducted by the agents. In addition, the shock might comprise a schedule of policy measures, which do not enter into force simultaneously.

It is straightforward to state necessary conditions in the context of opti-

[1] For the analysis of anticipated shocks with jumps in the state variables see Vind (1967) and Auernheimer and Lozada (1990).

mal control, which have to hold at points of anticipated shocks. This is the familiar requirement of continuity for adjoint variables, also known as the first Weierstrass-Erdmann corner condition. However, they do not imply that control variables are continuous. Since control variables are only required to be piecewise continuous, they can potentially jump twice, first at the point of the announcement, and second at the point where the parameter change actually takes place. Requirement of continuity for adjoint variables determines the height of the jump under weak regularity assumptions which we will state below.

In a second step, we describe how numerical computation can be conducted efficiently. We propose the relaxation algorithm as described by Trimborn et al. (2007) (in the following TKS) to simulate continuous-time, infinite horizon optimization models with preannounced shocks. The algorithm allows to solve expected shocks without any a priori information about the behavior of the model at the time of the parameter change. Only the dynamic system has to be provided together with the underlying parameters and their change along time. Moreover, the size of the jump in the control variables is calculated automatically. This is illustrated by employing the Ramsey-Cass-Koopmans model as a concise example.

The analysis of preannounced policy measures in the context of perfect foresight optimizing agents has a long tradition. The continuity of adjoint variables (or market prices like asset prices) has already been exploited in many cases (e.g. Judd, 1985, Howitt and Sinn, 1989, Turnovsky, 1996, Ch. 11). However, to our knowledge no formal derivation and analysis of this condition has been made in the economic literature. Moreover, no attempt has been made so far to construct algorithms which can solve these kind of problems in a generic way.[2]

In Section 2, we investigate the treatment of expected shocks in the context of optimal control theoretically. In Section 3, we describe the numerical

[2]Buiter (1984) describes how to solve the linearized problem.

implementation of expected shocks. In Section 4, we present the Ramsey-Cass-Koopmans model as a concise example. In Section 5 we conclude.

2 Theoretical investigation of expected shocks

Consider an agent who solves an infinite-horizon, perfect-foresight, continuous-time, optimal control problem[3]

$$\max_u \int_{t_0}^{\infty} f(t,x,u)dt \tag{1}$$
$$\text{s.t.} \qquad \dot{x} = g(t,x,u), \qquad x(0) = x_0$$

whereas x denotes a n_x dimensional state variable and u a n_u dimensional control variable. The functions f and g are, for the time being, assumed to be continuously differentiable functions in all three arguments. We assume that x is a continuous function of time, i.e. x cannot jump. On the other hand, u is only assumed to be a piecewise continuous function of time, i.e. u can exhibit interior jumps.

Consider a continuous change in a parameter or a variable exogenous to the agent. In problem (12) these parameters and variables are caught by the time arguments of f and g. That means a change in a parameter or exogenous variable along time causes a change of f and g along time. Due to perfect foresight the agent is aware of this time dependence.[4] Then, an anticipated shock in terms of a jump in a parameter after t_0 yields discontinuities in f and g.

For the analysis of expected shocks we follow Bryson and Ho (1975, pp. 101). First of all, the set of differential equations switches at, for the time being, an unspecified point of time $\tilde{t} > t_0$.

$$\dot{x} \;=\; g^{(1)}(t,x,u) \qquad t < \tilde{t}$$

[3]The following theoretical analysis as well as the numerical implementation is not tied to an infinite time horizon. However, in many cases infinite time horizon models are applied in marcoeconomics.

[4]However, the agent is not aware that potentially his decisions influence variables exogenous to him.

$$\dot{x} \quad = \quad g^{(2)}(t,x,u) \qquad t \geq \tilde{t}$$

Analogously, f changes its functional form at $\tilde{t}$ from $f^{(1)}$ to $f^{(2)}$. We fix the point of time where the jump in the parameter occurs with an interior boundary condition:

$$\psi[\tilde{t}] = \tilde{t} - t_{jump} = 0$$

Now, the transformed optimization problem is

$$\max_{u,\tilde{t}} \left[\int_{t_0}^{\tilde{t}} f^{(1)}(t,x,u)dt + \int_{\tilde{t}}^{\infty} f^{(2)}(t,x,u)dt \right] \tag{2}$$

$$\text{s.t.} \qquad \dot{x} = g^{(1)}(t,x,u) \qquad t < \tilde{t} \tag{3}$$

$$\dot{x} = g^{(2)}(t,x,u) \qquad t \geq \tilde{t}$$

$$x(0) = x_0 \tag{4}$$

$$\tilde{t} - t_{jump} = 0 \tag{5}$$

with t_{jump} denoting the (numerical) time value where the jump occurs. Note, that now $\tilde{t}$ has become part of the optimization problem. Necessary conditions for optimal solution are, employing Pontryagins Maximum principle (see Pontryagin et al., 1962) with the usual Hamiltonian $H = f + \lambda^T g$,

$$H_u = 0 \tag{6}$$

$$H_\lambda = \dot{x} \tag{7}$$

$$H_x = -\dot{\lambda} \tag{8}$$

together with the initial conditions (4) and transversality conditions[5]

$$\lim_{t \to \infty} \lambda(t)x(t) = 0\,. \tag{9}$$

Since g and f are defined piecewise before and after $\tilde{t}$, these conditions have to be applied piecewise. They have to be augmented by additional

[5]There is disagreement if this equation is indeed a necessary condition (see e.g. Chiang, 1992). We do not want to adress this point.

necessary conditions, which carry the information about the behavior at $\tilde{t}$. These conditions are derived from the interior boundary condition (see Bryson and Ho, 1975, pp. 101)[6]

$$\lambda(\tilde{t}^-) = \lambda(\tilde{t}^+) \tag{10}$$

$$H(\tilde{t}^-) = H(\tilde{t}^+) - \nu \tag{11}$$

whereas ν is a constant Lagrangian multiplier associated with the interior boundary condition. Note, that these equations require λ to be continuous, whereas H jumps at the particular point of time when the jump in the parameter takes place.

Equation (11) is the only equation where the multiplier ν appears. It equals the jump in the Hamiltonian at time $\tilde{t}$. Therefore, the set of equations and variables can be solved without equation (11), after which (11) trivially determines ν.

The problem at hand comprises $2 \cdot n_x$ differential equations and n_u algebraic equations, together with n_x initial conditions and n_x transversality conditions as well as $2 \cdot n_x$ interior boundary conditions. The latter comprise n_x conditions requiring x to be continuous and n_x conditions requiring λ to be continuous. Note first, that locally the dynamic behavior of the system can implicitly be described by $2 \cdot n_x$ differential equations in coordinates (λ, x) if H_u exhibits no singularity with respect to u.[7] For the system (2) to possess a unique solution it has to exhibit a n_x dimensional stable manifold around the stationary equilibrium or the stationary equilibria.[8] Applying Bellman's principle, we know that at time $\tilde{t}$ the solution has to be on this stable manifold. This results in a well defined two point boundary value

[6]Bryson and Ho (1975) explicitly address discontinuities in g but not in f. However, their proof also allows for discontinuities in f. Therefore, the extension is straightforward.

[7]This follows from the implicit function theorem. We assume the differential algebraic system to be of differential index one. Higher order differential algebraic systems exhibit a far more complex behavior (e.g. Ascher and Petzold, 1998, Ch. IV, 9). If $\frac{\partial H_u}{\partial u}$ exhibits full rank, the system is of index one.

[8]Some economic models exhibit a center manifold of stationary equilibria (e.g. Lucas, 1988). For transitional dynamics around this manifold see e.g. Hirsch et al. (1977).

problem for the time interval $[t_0, \tilde{t}]$ and for the time interval $[\tilde{t}, \infty)$. For both intervals, the $2 \cdot n_x$ differential equations together with n_u algebraic equations have to hold. The n_x initial boundary conditions for the first two point boundary value problem are (4) and the n_x final boundary conditions are represented by the requirement to be on the n_x dimensional stable manifold at $\tilde{t}$. For the second two point boundary value problem n_x initial boundary conditions are the requirement to start on the stable manifold, whereas n_x final boundary conditions are represented by the transversality conditions (9). Since initial and final boundary conditions of both problems are linked, this is labeled as a three point boundary value problem in the mathematic literature.

Statements about necessary or sufficient conditions for existence and uniqueness of solutions of two-point-boundary value problems are in no case as advanced as the theory of initial value problems (see Ascher and Petzold, 1998, Ch. III, 6). If the linearized system exhibits a unique solution it is at least possible to conclude that the original system exhibits a unique solution locally. However, it is beyond the scope of this paper to prove uniqueness of the linearized system. Necessary conditions for the existence of a unique solution comprise the dimension of the boundary conditions to add up to the dimension of the dynamic system. This is fulfilled here for both intervals, $[t_0, \tilde{t}]$ and $[\tilde{t}, \infty)$.

If a finite sequence of expected shocks is given, the same arguments as above can be applied. The resulting problem can be decomposed in a sequence of two point boundary value problems, each of which exhibits n_x initial and n_x final boundary conditions. Then the problem turns into a multi point boundary value problem.

3 Numerical implementation

There exists a rich economic literature how to solve infinite horizon maximization problems numerically (see TKS, Judd, 1998, or Brunner and Strulik, 2002 for a survey). Many approaches employ the first order conditions derived by Pontryagin's maximum principle for finding the solution, together with the initial conditions and transversality conditions (e.g. Judd, 1992, Brunner and Strulik, 2002, Mulligan and Sala-i-Martin, 1991, or TKS).[9] Then, the approach comprises the solution of a two point boundary value problem (e.g. Judd, 1992, and TKS), or the problem is transformed into a (stable) initial value problem (e.g. Brunner and Strulik, 2002 and Mulligan and Sala-i-Martin, 1991). However, the simulation of anticipated shocks transforms the two point boundary value problem into a multi point boundary value problem. Therefore, it is not straightforward to employ the existing solution algorithms. In the former case, the algorithms at hand cannot incorporate arbitrary internal boundary conditions without undergoing fundamental modifications. In the latter case, it is no longer possible to transform the problem into a single stable initial value problem.

In the mathematic literature it is suggested to reformulate the multi point boundary value problem into a two point boundary value problem to make it accessible for standard algorithms (see Ascher et al., 1985, and Ascher and Russell, 1981). However, we suggest to solve the multi point boundary value problem directly, which is straightforward if the relaxation algorithm as proposed by TKS is used. The advantage is that this proceeding is easy and intuitive, and it is not necessary to transform the system. The disadvantage is merely that the efficiency in terms of calculation requirement of the algorithm reduces slightly.

The special property of the multi point boundary value problem at hand is that the internal boundary conditions are fulfilled, if the variables λ and

[9] Exceptions are e.g. Mercenier and Michel (1994 and 2001), and Candler (1999).

x are forced to be continuous along the solution whereas no restrictions regarding u are made. The relaxation algorithm perfectly exploits this fact, since the algorithm exactly demands this property from differential and algebraic variables.[10] More precisely, the principle of relaxation is to replace the differential equations by approximate finite difference equations on a mesh of points in time. The residuum of the algebraic equations is minimized at every mesh point separately. Therefore, the relaxation algorithm treats differential and algebraic variables conceptually different. Whereas no connection along time is made for algebraic variables, differential variables are connected along time through difference equations. Therefore, algebraic variables can exhibit jumps, whereas differential variables have to be continuous but potentially can exhibit corners.[11] In case the solution indeed exhibits corners along the time path of the differential variables, the employed discretization rule decreases from second to first order. This means that by increasing the number of mesh points by factor x the global error reduces by x, while in case the solution does not exhibit corners the global error reduces by x^2.

In the economic literature, very often the set of algebraic equations is differentiated with respect to time and the adjoint variables λ are eliminated (see e.g. the analysis of the Lucas (1988) model by Caballe and Santos (1993) or Benhabib and Perli (1994)). In many cases, this eases economic interpretation, as well as numerical computation.[12] However, this is not advisable here, because differential equations for the control variables can only be applied piecewise. Since jumps can occur at internal points it would then be necessary to specify the height of each jump. This is, however, not possible without information about the adjoint variables. Therefore we

[10] We refer to differential variables as variables for which differential equations are present (i.e. λ and x) whereas the time derivative of algebraic variables does not appear in the set of equations (i.e. u).

[11] For a more precise description of the relaxation algorithm see TKS.

[12] Many algorithms cannot solve differential algebraic systems, and by eliminating the adjoint variables the dimensionality of the problem is kept small.

suggest to simulate system (6), (7), and (8) together with the boundary conditions and ensure that x and λ are continuous.

So far, we assumed that the differential algebraic system is scaled, i.e. the variables approach constant (finite) values in the long run. This may not be the case, for example if the model at hand is an endogenous growth model. For numerical computation the variables have to be scaled or detrended such that they approach constants in the long run. One possibility of scaling the system is to transform it into a system consisting of ratios of variables. If each ratio is constructed of variables that exhibit the same balanced growth rate, the resulting system of artificial variables is scaled. If ratios are created it has to be ensured that the time path of the differential variables remains continuous. That is, only combinations of different x and λ can be chosen for creating artificial differential variables. By contrast, arbitrary ratios can be created as algebraic variables. We recommend the scale adjustment as proposed by Lucas (1988), since continuous variables conserve their properties after scaling the system in this way.

To summarize, the relaxation algorithm can solve infinite horizon problems exhibiting anticipated shocks conveniently, since it solves the multi point boundary value problem directly. This is done by implementing the continuous variables as differential variables and the possible jumping variables as algebraic ones. The only input the user has to provide are the time dependent parameter values.[13]

4 A simple example

4.1 Description of the Model

To illustrate the numerical solution of infinite-time optimization models with preannounced shocks, we employ the Ramsey-Cass-Koopmans model (Ramsey, 1928; Cass, 1965; Koopmans, 1965) as an example. For reasons of clar-

[13] Using the supplemented software this can be done by stating an *if*-clause.

ity, we keep the model as simple as possible. We omit technological progress and assume the population to grow with constant rate n. We follow Barro and Sala-i-Martin (2004, Chapter 3) and introduce proportional taxes on wage income, τ_w, private asset income, τ_r, and consumption, τ_c. Thus the representative household's maximization problem is

$$\max_c \int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{(n-\rho)t} dt \tag{12}$$

$$\text{s.t.} \qquad \dot{k} = (1-\tau_w)w + (1-\tau_r)rk - (1+\tau_c)c - nk$$

whereas c denotes consumption per capita, k the capital stock per capita, w the wage rate, r the interest rate, σ the inverse of intertemporal elasticity of substitution, and ρ the discount factor, respectively.

The government is assumed to run a balanced budget. Therefore, government revenues equal total outlays. However, we assume that government spending appears nowhere else in the economy.[14] Firms produce according to a Cobb-Douglas production function.

$$Y = K^\alpha L^{1-\alpha}$$

whereas Y denotes the output, K the capital stock, L the amount of labor employed in production, and α the elasticity of capital in final-output production, respectively.

Since perfect competition in factor markets is assumed, firms pay the factors according to their marginal product.

$$\begin{aligned} r &= \alpha k^{\alpha-1} - \delta \\ w &= (1-\alpha)k^\alpha \end{aligned}$$

Turning back to the representative household optimization problem we now consider an anticipated change in the tax rates at time $\tilde{t} > t_0$. This means

[14] Alternatively, it could be assumed that government spending increases consumer utility, whereas households exhibit a additively separable utility function, or government revenues are spend on transfers to households. Since both alternative assumptions do not contribute anything to the topic discussed in this paper we chose the simplest assumption.

that at $\tilde{t}$ the consumer's budget constraint changes its functional form and potentially exhibits a jump on the right hand side.

$$\dot{k} = \begin{cases} (1-\tau_{w,1})w + (1-\tau_{r,1})rk - (1+\tau_{c,1})c - nk & t < \tilde{t} \\ (1-\tau_{w,2})w + (1-\tau_{r,2})rk - (1+\tau_{c,2})c - nk & t \geq \tilde{t} \end{cases}$$

For simplicity, we do not distinguish between $\tau_{i,1}$ and $\tau_{i,2}$ below, but denote the tax rates with a time index to indicate that different values have to be applied before and after $\tilde{t}$. Then, the Hamiltonian is[15]

$$H = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda((1-\tau_{w,t})w + (1-\tau_{r,t})rk - (1+\tau_{c,t})c - nk) \qquad (13)$$

Necessary conditions for an optimal solution are

$$c^{-\sigma} = \lambda(1+\tau_{c,t}) \qquad (14)$$

$$\dot{\lambda} = (\rho-n)\lambda - \lambda((1-\tau_{r,t})r - n) \qquad (15)$$

$$\dot{k} = (1-\tau_{w,t})w + (1-\tau_{r,t})rk - (1+\tau_{c,t})c - nk \qquad (16)$$

together with

$$\lambda(\tilde{t}^-) = \lambda(\tilde{t}^+) \qquad (17)$$

$$H(\tilde{t}^-) = H(\tilde{t}^+) - \nu\,. \qquad (18)$$

From the firm's sector we additionally have two algebraic equations for the factor prices, which we can substitute into the capital accumulation equation. Note, that for the derivation of the familiar Euler-Equation $\frac{\dot{c}}{c} = \frac{(1-\tau_{r,t})r-\rho}{\sigma}$ the time derivative of c must be taken. However, c is not necessarily differentiable at time $\tilde{t}$. Therefore, the Euler equation can only be applied piecewise.

For numerical computation of expected shocks we exploit the information that k and λ have to be continuous. Therefore, we implement them in the

[15] In contrast to the general derivation above we use the current-value Hamiltonian here, since then the shadow price λ is already a stationary variable. This does not affect conclusions about continuity, since H and λ only differ by the factor $e^{(\rho-n)t}$ with respect to the present-value Hamiltonian and its multiplier.

form of differential equations whereas we introduce c as an algebraic variable. To summarize, the differential algebraic system is

$$\begin{aligned}
\dot{k} &= (1-\tau_{w,t})(1-\alpha)k^{\alpha} + (1-\tau_{r,t})(\alpha k^{\alpha} - \delta k) - (1+\tau_{c,t})c - nk \\
\dot{\lambda} &= \lambda(\rho - (1-\tau_{r,t})r) \\
c^{-\sigma} &= \lambda(1+\tau_{c,t}) \, .
\end{aligned}$$

4.2 Simulation of expected shocks

We solve the model numerically employing the relaxation algorithm as described by TKS. Equation (19) displays that consumption will exhibit an interior jump if and only if the consumption tax τ_c evolves discontinuously, because λ must be continuous at the point of time of a preannounced shock. As an example, we will focus on an preannounced increase of τ_c from 10% to 20% at time $\tilde{t} = 20$. We assume the economy to be in steady state prior to the shock with a tax rate $\tau_c = 10\%$. At time zero the household experiences that at time $\tilde{t}$ the consumption tax will increase. Therefore, it will re-optimize its consumption plan such that consumption potentially jumps immediately as well as at time $\tilde{t}$. We conduct three simulations with the inverse of intertemporal elasticity of substitution, σ, equal to $\frac{1}{2}$, 1, and 2, respectively.[16] For the simulation no a priori information about the time path of the variables or the shape of the flow is given. We choose, as an initial guess, all variables to be constant at their steady state values. This always leads to quick convergence.

Note, that the steady state value of k is not affected by a change in τ_c while the steady state value of c is reduced by $\frac{1+\tau_{c,\tilde{t}-}}{1+\tau_{c,\tilde{t}+}}$. If the tax on consumption increases, households consume less in the long run while spending the same amount for consumption.[17] Therefore, it is straightforward to ana-

[16] The remaining parameters and tax rates are set to $\alpha = .3$, $\delta = .03$, $\rho = .02$, $n = .01$, $\tau_w = .4$, and $\tau_k = .3$, respectively.

[17] Households spend $c(1+\tau_c)$ for consuming the amount c.

lyze an unexpected, immediate shock at t_0. In this case, consumption would jump to its lower, new steady state value at t_0 without any dynamics in capital. This analysis holds for any feasible set of parameters.

For the case of an anticipated shock we can reason that consumption jumps down at the preannounced time of the tax increase, $\tilde{t}$. It is not optimal for the household to smooth consumption such that this jump vanishes, which again can be seen from equation (19).

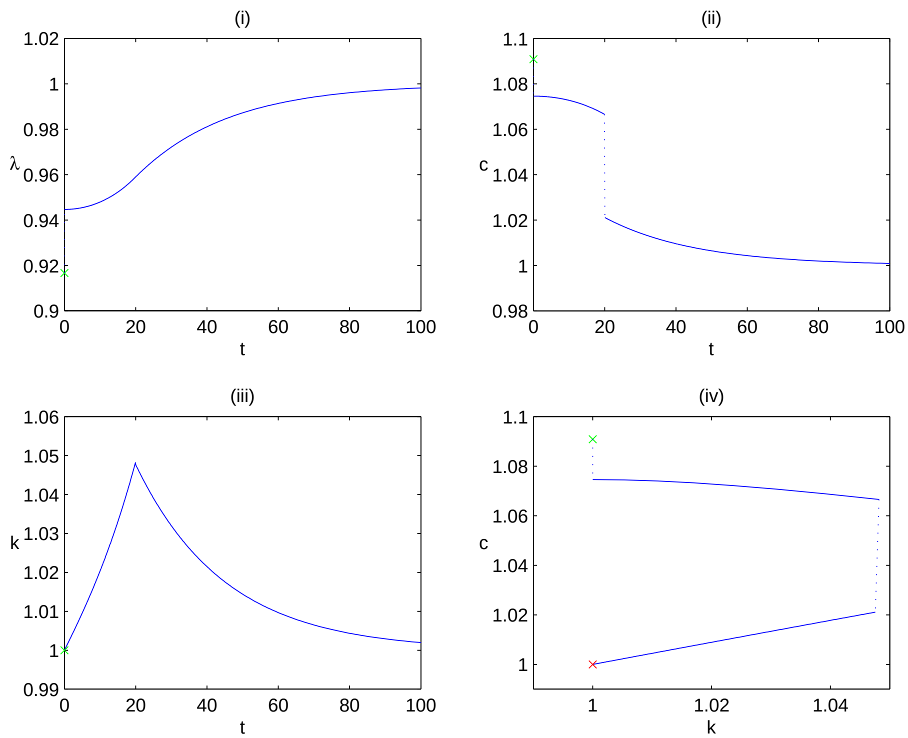

Figure 1: Anticipated increase in τ_c with $\sigma = 2$

If σ is high, i.e. $\sigma = 2$, consumers have a strong preference for smoothing consumption over time. Figure 1, (i), (ii), and (iii) display the time path of λ, c, and k, respectively, referring to this simulation. Variables are normalized to unity at the new steady state, whereas green crosses designate old steady state values. Households try to soften the drop in consumption at $\tilde{t}$. They do so by abandoning consumption in the period between t_0 and $\tilde{t}$. Figure 1, (ii) shows that consumption drops down immediately at t_0 and is decreasing in the subsequent period until $\tilde{t}$. Then, it jumps down again and is approaching the new steady state from above. Households increase savings until $\tilde{t}$, and dissave after $\tilde{t}$, which can be seen in Figure 1, (iii). Figure 1, (iv) shows the (c, k)-phase diagram. The economy starts initially at the green cross and moves along the blue line to the new steady state indicated by the red cross. Dotted lines designate jumps, whereas solid lines designate continuous dynamics along time. Note that λ is continuously differentiable whereas k is continuous but experiences a corner at $\tilde{t}$ (Figure 1, (i) and (iii)).

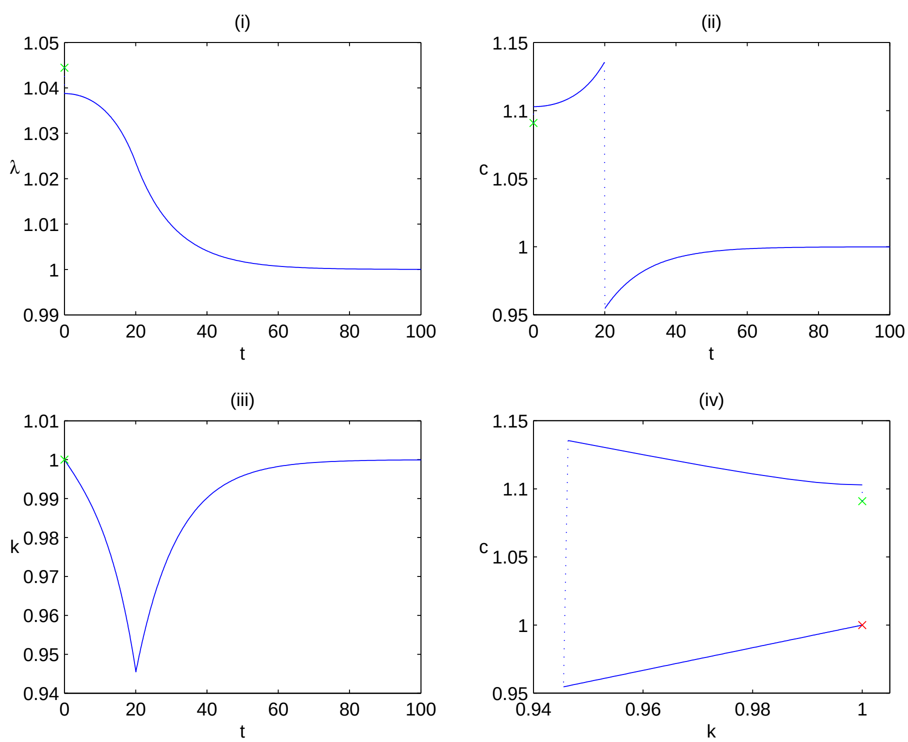

Figure 2: Anticipated increase in τ_c with $\sigma = .5$

Conversely, if σ is low, i.e. $\sigma = 0.5$, households do not put much emphasis on consumption smoothing. They are willing to accept sharp kinks in consumption if this on the other hand yields initially higher consumption. Figure 2, (i), (ii), and (iii) display the time path of λ, c, and k, respectively, referring to the second simulation. Again, variables are normalized to unity at the new steady state, whereas green crosses designate old steady state values. In Figure 2, (ii) it can be seen that consumption jumps up at t_0 and increases in the subsequent period until $\tilde{t}$. Then, it jumps down and is approaching the new steady state from below. Households, therefore, exploit the fact that for the same amount spent for consumption they receive a higher consumption before $\tilde{t}$ than after. They do so by dissaving before $\tilde{t}$ (Figure 2, (iii)) and regain their former level of assets by saving after $\tilde{t}$. Figure 2, (iv) shows the (c, k)-phase diagram. The economy starts initially at the green cross and moves along the blue line to the new steady state indicated by the red cross. Again, dotted lines designate jumps, whereas solid lines designate continuous dynamics along time. As in the first simulation, λ is continuously differentiable whereas k is continuous but experiences a corner at $\tilde{t}$ (Figure 2, (i) and (iii)).

For $\sigma = 1$, which implies a logarithmic utility function, a third pattern emerges. Then, consumption does not jump at t_0 and remains in its old steady state value. At $\tilde{t}$ consumption jumps down by $\frac{1+\tau_{c,\tilde{t}-}}{1+\tau_{c,\tilde{t}+}}$ and the economy is immediately in its new steady state.[18] The model's behavior is as if households would experience an unexpected shock, since the above mentioned counteracting effects cancel each other.

This concise example demonstrates that first of all control variables can exhibit interior jumps. In this model, the optimal path of consumption for households jumps at the time of the preannounces shock. To determine the height of the jump the adjoint variables are necessary and cannot be elimi-

[18]We do not show simulation results for this parameter setting since all variables are constant, despite from negligible numerical errors.

nated from the system of differential equations. Second, it is demonstrated that accounting for the preannounced component of shocks can bring additional theoretical insights, which cannot be derived from a stylized shock. In this model, an analysis of the stylized shock did not show any dynamics. Note also, that the point of the stable manifold, which the economy hits at time $\tilde{t}$, cannot be derived from the phase diagram, but from numerical computation.

5 Conclusion

We derived the well-known continuity principle for adjoint variables for preannounced or anticipated changes in parameters for a broad class of infinite-horizon, perfect foresight, optimization models. For easy and intuitive numerical computation of the resulting multi point boundary value problem we suggested to simulate the resulting differential algebraic system representing the first order conditions. By ensuring that the state variable x and the adjoint variable λ are continuous, potential jumps in the control variable u are calculated automatically. This can be easily conducted with the relaxation algorithm as proposed by TKS. This algorithm treats differential and algebraic variables conceptually different such that the requirements for simulating the multi point boundary value problem at hand are automatically met.

The proposed algorithm was employed to solve a Ramsey model extended by an elementary Government sector. Simulations of a preannounced increase in the consumption tax showed a qualitative different pattern depending on the intertemporal elasticity of substitution.

Potential applications of this method emerge throughout in economic fields where the reaction on preannounced policy measures is of special interest in the context of perfect foresight optimizing agents.

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