Energy Distribution, Power Laws, and Economic Growth

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Abstract. The natural sciences have established a general scaling law that relates metabolism and body size of animals. Recently this association – known as Kleiber's law – has received deep theoretical foundation by network theory and has been fruitfully applied to explain various biological phenomena, in particular ontogenetic growth. Here we derive a similar power law for economic metabolism (energy consumption per capita) and economic size (capital per capita). Invoking the power law we provide a metabolic-energetic founded law of motion for capital per capita. Using data for the U.S. states we test the resulting structural model and find evidence in favor of a scaling parameter, between energy and capital per capita, of about 2/3.

Keywords: Economic Growth, Energy, Metabolism, Power Laws, Networks.

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Ever since Kleiber (1932) it has been known to biologists that a strong correlation, referred to as "Kleiber's law", is found between energy intake of biological organisms and their body mass (*1*). The two are scaled according to a power law $B = B_0 \cdot m^b$, where B is basal metabolism, m is mass, B_0 is a constant, and $b = 3/4$. It has been found that the association holds across biological systems spanning more than 20 orders of magnitude in mass; from the molecular level up to whales (*2*). Recently, biologists and physicists in collaboration have started to provide microfoundations. The common denominator of these theories is that they seek to explain the power law as a manifestation of how energy is diffused and absorbed in biological systems, viewed as energy transporting networks (*3–6*).

Power laws are not unfamiliar to economists although less frequently employed than by the natural sciences (*7, 8*). Recently it has been shown that in particular city size correlates with a variety of social, economic, and biological variables according to power laws (*9, 10*). While many of these interesting associations are still theoretically unexplained they substantiate empirically that economic entities can be seen as "organisms" and accordingly economic activity as "metabolism" (*11–14*).

Here we take up this view with a special focus on capital accumulation and economic growth. Our central hypothesis is that an important enabling factor behind the process of capital accumulation is the ability of societies to distribute energy to the sites where the use of capital, understood as energy consuming machines and appliances, is deemed worthwhile. We demonstrate a power law association between energy consumption per capita and capital per capita of a geographic entity (country, state, or region). Together with an accounting equation stating that energy is used to create, run, and maintain capital we get a law of motion for capital akin to the reduced form of the Solow model (*15*). Our empirical analysis, invoking data for U.S. states, delivers strong support for the energy-founded model.

1. The Economy as a Network

In biology the power law is derived from the notion of living organisms as energy transporting networks whereby the size of the organism is given by the number of energy consuming units, i.e. the number cells that have to be fed (*3–6*). By way of analogy we view an economic "organism" as an energy transporting network. The purpose of the network is to deliver (non-human) energy to the energy-consuming units of the economy. Specifically, energy is assumed to originate from a source (a power plant), and is diffused across the economy via a power grid to the sites at which it is used. In keeping with the established terminology (*5*), we refer to each site as "a transfer site"; this is where energy is converted into work effort. Each transfer site is assumed to be scale-invariant. A reasonable way to think about the transfer sites is as electricity outlets, which can be said to fulfill this requirement. Moreover, all transfer sites are locally connected, and thus linked to the source either directly or indirectly via transmission lines.

The size of the network is defined by the geometric size of the shape which defines its outer contours. In biology the outer contour of the network is tangible - the body. In the present case it is abstract; i.e., the geometric shape which *would* be able to engulf the network. The fact that this geometric shape is not tangible is immaterial for the argument.

Let the linear size of the network be denoted by L so that the total size of the network is proportional to L^D, where D is the dimension of the network. The distance between a transfer site and the source i is defined by the number of transfer sites one will have to pass to get from i to the source. Because the network expands (if it expands) in an outward direction, it is inevitable that the mean distance between the transfer sites and the source rises as the network becomes larger. The specific nature of the network matters for how large the increase is (*5*).

Generally, assuming a space-filling network and scale-invariant transfer sites, the number of transfer sites must rise with the size of the network. As each transfer site uses energy, one may anticipate an association between the size of the network and the total energy consumed at the transfer sites, E. More specifically, we assume that for a given size of the network L^D total energy consumption is (log-) linear in population size P.

$$E \propto L^D \cdot P. \tag{1}$$

As a result, a change in *per capita* energy consumption requires a proportional change in the size of the network, $e \propto L^D$. That is, per capita energy consumption, e, is ultimately attributable to the number of devices (e.g. television sets, washing machines, computers and so on), which a given population utilizes. The notion is that every time a new piece of equipment is connected to an electricity outlet, a new transfer site emerges, and the network expands allowing for more energy consumption per capita.

Empirically, strong support has been found for such a linear association between E and P for a cross section of German cities (*9*) and Chinese urban administrative units (*10*). In the empirical section we provide additional support using cross-state data for the US.

A key result in Banavar et al.'s (*5*) network theory is the proof of an association between *total* flow of energy in a network, F, and the size of the network given by $F \propto E \cdot L^x$, where x depends on the efficiency of the network. Specifically, $x = 1$ in directed networks, which minimize total energy requirements needed to fuel the economy (or the organism in biology) subject to the requirement that all sites are served. In the most inefficient network, the space-filling spiral, $x = D$. Inserting (1) we get $F \propto L^{D+x} \cdot P$. In other words, when the size of the network rises the energy flow per capita (F/P) expands at least in proportion to L^{D+1}, and at most in proportion to L^{2D}.

Finally, we assume proportionality between the total capital stock and the total energy $F \propto K$. This assumption is thought to capture that capital is nested at the transfer sites *and* in the network itself, in the form of the transmission lines. Hence capital is needed to transfer energy (and "hosts" energy in the process) to the sites where capital uses energy. Energy conservation in the system at large (at any given instant in time) would then suggest proportionality between the capital stock and the total flow of energy in the system.

2. A Power Law for Energy Consumption of Economies

The established associations between capital stock, size of the network, and energy consumption per capita, $F \propto K$, $F \propto L^{D+x} \cdot P$, $e \propto L^D$ can be summarized as a log-linear association between energy use per capita and the amount of capital per capita k, determined up to a constant ϵ. It represents the reduced form of the economy as a network:

$$e = \epsilon k^a, \qquad a \equiv \frac{D}{D+x}. \tag{2}$$

The intuition for the concave scaling association is the following. When K rises new transfer sites emerge, and the size of the network expands. As a result, the mean distance between the source and the transfer sites increases. A greater mean distance between the sites and the source implies that a greater fraction of total energy supply (F) is used to "fuel" the system, as opposed to being available for consumption at the sites (E). Accordingly, the concavity reflects the difficulty in delivering increasing amounts of energy to machines connected to the power grid, when the size of the network expands. This will ultimately be the reason why the process of capital accumulation is bounded, absent technological changes, as shown below.

Notice that the scaling coefficient, a, should fall in a $[1/2,\ \ D/(D+1)]$ interval. For a more precise prior, we need to pin down D. The most natural notion is probably that $D = 3$, i.e. a three-dimensional network. In this case the scaling exponent a should fall in the interval $[1/2,\ \ 3/4]$, depending on the efficiency of the system.

3. A Model of Capital Accumulation and Energy Consumption

Energy is used to run, maintain, and create capital. Here we use a notion of capital that is broader than the national accountants' definition. It includes all energy consuming appliances. That is, the k appearing here also includes durable consumption goods. As a result, we do not distinguish whether, for example, an air-conditioning system is placed in a firm or a private household.

Assume time is continues, and let μ be the energy requirement to operate and maintain the generic capital good while ν is the energy costs to create a new capital good. In that case energy conservation implies $E(t) = \mu K(t) + \nu \dot{K}(t)$. For future reference, observe that if we were to shut off energy use, the capital stock would be declining over time at the rate μ/ν, due to lack of maintenance and replacement. Hence, the ratio μ/ν captures the physical phenomenon of *capital depreciation*.

Dividing through by population size P in the energy conservation equation we get

$$e(t) = \mu k(t) + \nu \frac{\dot{K}(t)}{P(t)}.$$

Assume that the population grows at a constant rate of n. Then $\dot{K}(t)/P(t) = \dot{k}(t) + nk(t)$. Inserting this and the power law association (1) into the above equation provides the law of motion for economic growth:

$$\dot{k}(t) = \frac{\epsilon}{\nu} k(t)^a - \left(\frac{\mu}{\nu} + n\right) k(t). \tag{3}$$

Economic growth (of the total capital stock) according to (3) is structurally identical to ontogenetic growth (of body cells) as derived by West et al. (*16*). Furthermore, the above equation is also structurally identical to the reduced form of the Solow model, the cornerstone of standard economic growth theory (*15*).

The standard Solow model builds essentially on three elements: the income identity of a closed economy, an aggregate (Cobb-Douglas) production function, and the assumption of a constant savings/investment rate. If income is invested it is turned into new units of capital, which elevates income (via the production function) and enables further accumulation. The process is bounded by the concavity of the production function, reflecting diminishing returns to capital input.

A similar process can, as shown above, be based on the assumption of energy conservation and a network vision of the economy. Since we do not distinguish between capital used in production (the national accounts notion of "capital") and capital used in consumption (durable consumption goods, in national accounts), the savings/investment rate does not play a particular role. The present model is to be viewed as characterizing the evolution of energy consuming goods that it is feasible for an economy to sustain; the division of such goods for "consumption purposes" and "production purposes" is ignored.

Observe that we can obtain a *dual representation* of the model in terms of energy consumption, by using $e = \epsilon \cdot k^a$ in (3):

$$\dot{e}(t) = \left[\frac{\epsilon^{1/a}}{\nu} \cdot e(t)^{-(a-1)/a} - \left(\frac{\mu}{\nu} + n \right) \right] \cdot a \cdot e(t) \tag{4}$$

Accordingly, this equation provides a description of the evolution of energy consumption per capita over time. This representation will prove useful in the empirical analysis below.

4. Discussion

The model above has similar formal properties to the Solow model. In particular, there exists an equilibrium, or steady-state, to which the economy adjusts; the steady state is unique and globally stable.

The adjustment process works as follows. At any given instant in time k is predetermined. Given k a size of the underlying network is implied, and consequently the supply of energy (which is assumed to adjust), e, is determined. If e is sufficiently large, i.e. it exceeds the energy needs required to maintain and run existing capital, the stock of capital can expand further. However, as the network expands the amount of additional energy which can be made available for direct use starts to diminish (i.e. E/F declines). Eventually, therefore, the system settles down at a steady state level of $k = k^* \equiv [(\mu + n\nu)/\epsilon]^{1/(a-1)}$.

For a proper assessment of the steady-state result note that it is not derived from the assumption of limited supply of energy. Instead, convergence towards a constant capital stock per capita is a consequence of energy demand, distribution, and the entailed decreasing returns from expanding networks. It is sometimes argued that economic growth is ultimately limited from above by energy availability (*17*). Interestingly, however, the present analysis demonstrates that – absent technological progress – economic growth is limited *even if* energy supply were unlimited. This brings us to the issue of how "technology" is said to be present in the model above.

In the energy-based model of capital accumulation technological progress manifests itself as a permanent improvement in the use and distribution of energy (i.e. a parameter change of ϵ, ν, μ, or a). Economic development is thus understood as a perpetual series of efficiency gains in appropriating non-human energy. This way we can think of, for example, the wheel (the wheeled plow) as a device that exploits kinetic energy more efficiently and of the system of three field crop rotation as a device that exploits solar energy more efficiently than previously available methods. In the model it could be (crudely) captured by, for example, an increase in ϵ. Likewise, the discovery of how to harness electricity, and the construction of the power grid, has improved the use and diffusion of energy in a historically unprecedented way. For the first time in human history man could accumulate energy-requiring capital without constraints related to the presence of water or wind. Effectively, it can be viewed as the genesis of the network equation. Similarly, to give a final example, one would expect Nano-technology to increase the scope for capital accumulation yet again. For example, by allowing for more energy to be transported through the network (due to less power losses), which could be represented by an increase in a and ϵ, or a reduction in μ or ν.

The model allows some reconciliation between neoclassical growth theory and the work of its staunchest critics (*17, 19*). The central charge is that energy is introduced into the standard models in an unsatisfactory way (if not ignored altogether). That is, by including energy in the aggregate production function as a separate input, which can be substituted for by capital. This approach is fundamentally flawed, the argument goes, because it does not consider that any capital good is itself produced by means of energy. Here, we have explicitly taken into account that all capital is created, run, and maintained through energy use. Interestingly, we nevertheless arrive at a law of motion for capital which is structurally identical to that implied by the Solow model. Hence, the structure of the Solow model is not at variance with fundamental physical principles, like energy conservation.

5. Empirical Evidence

The purpose of this section is to obtain an estimate for the scaling parameter a; if a is above 3/4, or below 1/2, we can reject the model as it stands.

As a preliminary test, however, we test the validity of the crucial underlying assumption of the model, the postulated linear association between total electricity consumption and total population (1). It is based on the following specification

$$\log(E_i) = a_0 + a_1 \cdot \log(P_i) + \varepsilon_i, \tag{5}$$

where i denotes the unit of observation. The variable ε_i is a noise term. That is, structurally $L_i^D = a_0 + \varepsilon_i$, which allows the size of the network to vary across units of observation. Our assumption, for which we require validation, is that $a_1 = 1$. The identifying assumption, for ordinary least squares (OLS) to deliver an unbiased estimate for a_1 is that $E(\varepsilon \cdot P) = 0$, where $E(\cdot)$ is the expectation operator. In terms of the model, this means that the size of population is uncorrelated with the capital-labor ratio, since the latter determines L. Under the null (the model is correct), this assumption is plausible.

We obtained data for electricity sales across the U.S. states from the US energy information administration. Specifically, electricity sold to end users: the residential sector, the commercial sector, and the industrial sector. The data is available free of charge at http://www.eia.doe.gov/emeu/states/ _seds.html. From this source we also obtained data for state populations.

Figure 1 about here

Figure 1 provides a visual impression of the results from estimating equation (5) by OLS and reports the parameter estimates; the regression relates to the year 2000.

As is visually obvious, the cross-state sample corroborates previous findings for German cities (*9*) and Chinese urban administrative units (*10*). The coefficient for $\log(P)$ is close to 1, with a fairly narrow 95% confidence interval. Hence, Figure 1 informs us that a potential empirical failure of the theory cannot be ascribed to a violation of equation (1).

The second test concerns the ultimate structure of the model. One possibility would be to test the "primal" equation (3). Unfortunately, data poses a problem. While it is feasible to obtain estimates for the capital stock from national accounts statistics by employing the perpetual inventory method (i.e., by cumulating investments), we require a broader notion of capital, as mentioned above. Hence a different approach is needed to test the model, and estimate a. Fortunately, instead of estimating the primal model we can use the fact that the model has a dual representation (4) in energy consumption.

In order to facilitate estimation we log-linearize (4) around the steady state where $e = e^* \equiv \epsilon[(\mu + n\nu)/\epsilon]^{a/(a-1)}$:

$$\ln[e(t)] = \ln[e(0)]\,e^{-\lambda t} + \left(1 - e^{-\lambda t}\right)\left[\frac{1}{1-a}\ln(\epsilon) - \frac{a}{1-a}\ln(v) - \frac{a}{1-a}\ln(n + \mu/\nu)\right],$$

where $\lambda \equiv (1-a)(n + \mu/\nu)$. Because the equation is non-linear in the parameter of interest (a), it cannot be estimated by way of ordinary least squares. Instead we will have to resort to an iterative procedure. Below we therefore report the results from estimating a by non-linear least squares (NLS).

Some of the variables entering the estimation equation are unobservable: the energy costs of running and maintaining capital (μ), the costs associated with creating new capital (ν), and the efficiency parameter ϵ. From the primal model, however, we have a prior for the *ratio* μ/ν; it should reflect the rate of capital depreciation. Hence, a reasonable number for μ/ν can be imposed from the literature; a typical finding for the US is a rate of capital depreciation of 6% (*20, 21*), which we therefore impose *a priori*.

In sum, the equation we estimate is:

$$\ln[e_i(T)] - \ln[e_i(0)] = -\left[1 - e^{-b_1\cdot(n_i+0.06)\cdot T}\right][b_2 + b_3\ln(n_i + 0.06) + \ln e_i(0)] + u_i, \quad (6)$$

where the predictions of our theory are: $b_1 > 0$, $b_3 > 0$ and $1 - b_1 = b_3/(1+b_3) = a \in (1/2, 3/4)$. The parameter b_2 captures the influence from ϵ and ν; its constancy reflects the belief that these variables are constant across the 50 states. We find this identifying assumption plausible. The parameters ν, ϵ and μ are arguably of a technological nature, as discussed above, and the U.S. states are undoubtedly fairly homogenous in this respect. Omitting these parameters is therefore likely to be only a minor problem.

We test the model over the period 1960-2000; data on n and e derive from the US energy information administration. Since we are examining the period 1960-2000, $T = 40$; u_i is a noise term, which is assumed to be uncorrelated with the right hand side variables. The results from estimating equation (6) are reported in Table 1.

- Table 1 about here -

In panel A we report the results from estimating the model freely. That is, without imposing the restriction $1 - b_1 = b_3/(1 + b_3)$. As can be seen from the R^2, the model accounts well for the cross-state variation in log changes in energy consumption per capita over the 1960-2000 period. Moreover, the signs of the parameters are as predicted: $b_1 > 0$, $b_3 > 0$; both are significant at the 1% level of significance. In contrast, the parameter for b_2 comes out insignificant. Taken at face value this suggests $\frac{1}{1-a}\ln(\epsilon) = \frac{a}{1-a}\ln(v)$. The *structure* of the model is not rejected by the data either. That is, the implied equality of $1 - b_1$ and $b_3/(1 + b_3)$ is not rejected.

Consequently, we also estimated the model where $a = 1 - b_1 = b_3/(1 + b_3)$ is imposed. The results are shown in Panel B. The point estimate of the scaling exponent a is 0.62; close to 2/3. Interestingly, the confidence interval for a coincides almost perfectly with the predicted range for a, as suggested by theory (i.e. $a \in [1/2,\ 3/4]$). As a result, we can reject a being smaller than the lowest admissible value under the theory: 1/2. At the same time, "full efficiency" can be rejected: $a < 3/4$ at the 5% level. In biology it is reasonable to argue that nature ensures full efficiency, by way of natural selection, which implies a scaling coefficient between basal metabolism and body size (in theory (*3–6*)) of 3/4 . In the context of man-made networks it is perhaps unsurprising to find evidence in favor of some inefficiencies.

In sum, the above results support a model of energy consumption which builds on two elements: (i) Log-linear scaling between capital per capita and electricity consumption per capita, and, (ii) an accounting equation relating electricity consumption to capital use, maintenance and accumulation. Moreover, the state-level data suggests that a scaling coefficient around 2/3 is the best approximation.

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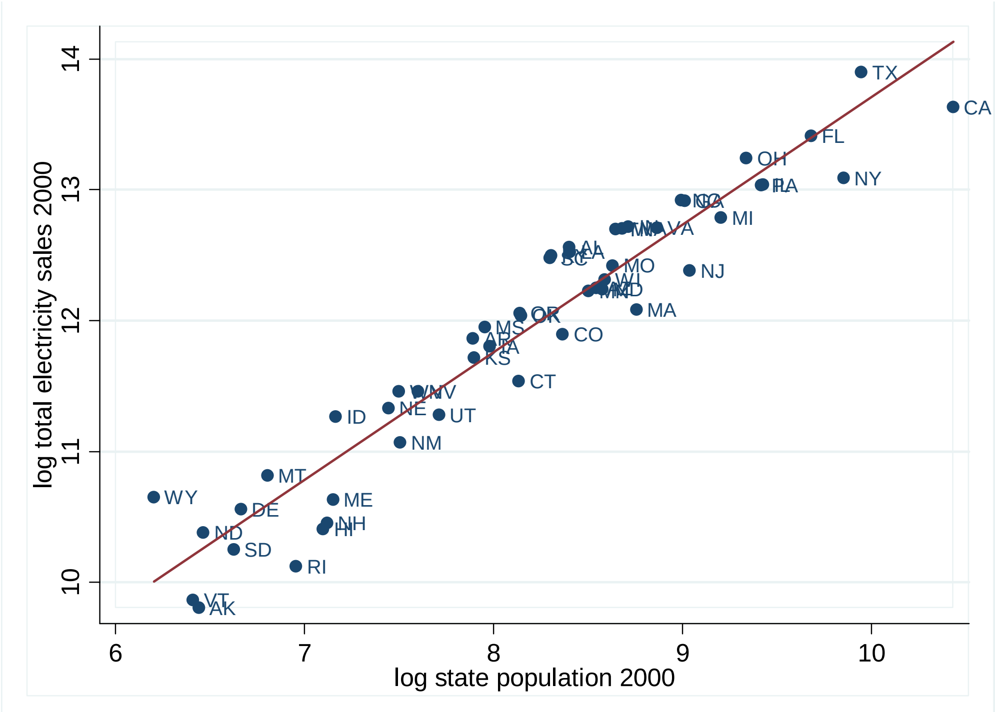
Figure 1: Population Size and Energy Consumption Across US States



Electricity consumption vs. Population size: 50 US States in 2002. Note: The line illustrated is estimated by OLS, and yields the following result: log(Electricity) $= 3.95 + 0.98 \cdot$ log(Population), $R^2 = 0.92$, 95%, CI for population coefficient is (0.87,1.08), Observations =50.

Table 1. NLS estimates of the model

Panel A Unrestricted Model	
Dependent variable: log change in energy consumption per capita 1960-2000	
b1	0.36 (0.07)
b2	-0.38 (0.89)
b3	1.55 (0.37)
1-b1=(b3/(1+b3) (p-value)	0.60
R^2	0.96
Obs.	50
Panel B Restricted model; 1-b1=(b3/(1+b3) imposed	
1-b1	0.62 (0.05)
b2	-0.05 (0.88)
95 % CI for a	(0.52,0.73)
R^2	0.96
Obs.	50
Notes: Heteroskedasticity robust standard deviations in parenthesis.	