Evaluating a class of nonlinear time series models[*]

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Abstract

We consider a recently proposed class of nonlinear time series models and focus mainly on misspecification testing for models of such type. Following the modeling cycle for nonlinear time series models of specification, estimation and evaluation we first treat how to choose an adequate transition function and then contribute to the evaluation stage by proposing tests against serial correlation, no remaining nonlinearity and parameter constancy. We also consider evaluation by generalized impulse response functions. The finite sample properties of the proposed tests are studied via simulation.
We illustrate the use of these methods by an application to real exchange rate data.

JEL-Numbers: C12, C22, C52

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1 Introduction

Over the last decade and a half the interest in nonlinear time series models has been grown steadily. Especially the class of smooth transition autoregressive (STAR) models, initiated by the work of Bacon and Watts (1971) and popularized by Teräsvirta (1994), has enjoyed great success. A lot of work in this area has been devoted to estimation, specification, testing and

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applications such as forecasting. For a recent review of this field see e.g. Potter (1999) and van Dijk et al. (2002b).

Within the class of STAR models the two most notable are the logistic STAR (LSTAR) and the exponential STAR (ESTAR) model. The ESTAR model in particular has been very popular with empirical investigations of economic theories such as purchasing power parity (PPP) or the Fisher hypothesis (see e.g. Taylor et al. (2001) and Rose (1988)). It is by now well known that the estimation of the parameters in ESTAR models is notoriously hard as noticed early by Haggan and Ozaki (1981), Tong (1990) or Teräsvirta (1994). In fact, as shown by Donauer et al. (2010) some crucial parameters in ESTAR models are unidentified if the variance of the innovation term becomes very small. This is especially important within the context of real exchange rates as estimated innovation variances are usually extremely small (see e.g. Gatti et al. (1998), Öcal (2000), Taylor et al. (2001) or Rapach and Wohar (2006) among others). In order to remedy this problem Donauer et al. (2010) propose a new type of nonlinear model formulation called T-STAR that maintains the desirable properties of ESTAR but reduces the estimation problem for the most part. In addition the authors propose a linearity test and an unit root test for the new model.

A complete account of this new model however would require to run through all steps of the empirical modeling cycle devised by Teräsvirta (1994). This paper is mainly devoted to the evaluation stage of the modeling cycle and proposes a suite of Lagrange multiplier (LM) tests designed for this newly developed model. The development of specialized parametric tests for nonlinear models is important since standard misspecification tests such as the well known Ljung-Box test have been shown to be badly sized when the true data generating process is nonlinear (see Eitrheim and Teräsvirta (1996)). However, before considering residual based tests we treat the problem of how to choose between two different nonlinear models by proposing a direct test based on the encompassing principle.

The paper is organized as follows: In section 2 we review the class of models under study and the empirical modeling cycle for nonlinear models. A method to discriminate between competing nonlinear model formulations is described in section 3. In section 4 we propose different tests against serial dependency of the residuals, no remaining nonlinearity of the residuals as well as parameter constancy of the estimated model. In addition we cover evaluation via impulse response analysis. In section 5 we study the finite sample performance of the afore mentioned tests. In section 6 we run through the whole modeling cycle to model real exchange rates before section 7 concludes. Some additional results are collected in the appendix A.

2 The modeling cycle

A general STAR model is given by two autoregressive regimes connected by a smooth transition function. Smoothness means that the transition function changes continuously from zero to one and therefore governs the transition between the two regimes in a smooth way. Alternatively, a STAR model can also be interpreted as a continuum of regimes which is passed through by the process.

In general, univariate STAR(p) models, $p \geq 1$ and $d \leq p$, are given by

$$y_t = [\Psi w_t] \times [1 - \mathcal{G}(y_{t-d}; \gamma, c)] + [\Theta w_t] \times \mathcal{G}(y_{t-d}; \gamma, c) + \varepsilon_t \tag{1}$$

$$= [\Psi w_t] + [\Phi w_t] \times \mathcal{G}(y_{t-d}; \gamma, c) + \varepsilon_t, \quad t \geq 1, \tag{2}$$

with $\varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$.

The parameter vectors Ψ and Θ as well as w_t are given by $\Psi = (\psi_0, \psi_1, \ldots, \psi_p)$, $\Theta = (\vartheta_0, \vartheta_1, \ldots, \vartheta_p)$, and $w_t = (1, y_{t-1}, \ldots, y_{t-p})'$. For the alternative parametrization in (2) we have $\Phi = (\varphi_0, \varphi_1, \ldots, \varphi_p) = (\psi_0 - \vartheta_0, \psi_1 - \vartheta_1, \ldots, \psi_p - \vartheta_p)$, i.e. the second regime realizes as sum of Ψ and Φ.

Different choices of the transition function $\mathcal{G}(\cdot; \gamma, c) : \mathbb{R} \to [0, 1]$ lead to different STAR models. A popular choice is the exponential form leading to the ESTAR model

$$\mathcal{G}(\cdot; \gamma, c) = 1 - \exp\left\{-\gamma (y_{t-d} - c)^2\right\}; \quad \gamma > 0 . \tag{3}$$

This functional form for the transition function is popular for modeling real exchange rates or real interest rates (see e.g. Kapetanios et al. (2003)). However the estimation of the parameter γ that governs the functional form and thereby the transition speed is notoriously hard. Indeed it is shown by Donauer et al. (2010) in Theorem 2.4 that

$$\lim_{\sigma_{\varepsilon_t} \downarrow 0} \text{Var}(\hat{\gamma}) \to \infty . \tag{4}$$

As the parameter γ is absolutely crucial in STAR models (see Tong (1990)) this result makes it almost impossible to obtain reliable estimation results for reasonable small innovation variances.

To remedy this problem Donauer et al. (2010) propose to reformulate the well known transition function in (3) as

$$\mathcal{G}(\cdot;\gamma,c) = 1 - \left\{1 + (y_{t-d} - c)^2\right\}^{-\gamma}; \quad \gamma > 0\,. \tag{5}$$

The resulting model is called T-STAR model and reduces the identification problem for the most part. Both transition functions in (3) and (5) share the same properties which make them applicable in the same situations. For further details we refer to Donauer et al. (2010).

The modeling cycle for nonlinear models as proposed in Teräsvirta (1994) consists of three main steps:

Step 1: Specification

- Specifying a linear autoregressive model via an information criterion such as AIC (see Akaike (1974)) or BIC (see Schwarz (1978)).
- Testing linearity for different values of d and if it is rejected specify d by minimizing the p-value of the linearity test via a grid search over possible values of d (see Tsay (1986) or Teräsvirta (1994)).
- Choosing an adequate transition function by testing a series of nested hypotheses (see Teräsvirta and Anderson (1992), Teräsvirta (1994) and Escribano and Jordá (2001)).

Step 2: Estimation

- Estimate the specified model using either nonlinear least squares or conditional (quasi) maximum likelihood. Consistency for these techniques has been established by Klimko and Nelson (1978) and Tjøstheim (1986) respectively.

Step 3: Evaluation

- Perform residual based tests against serial dependence, no remaining nonlinearity and parameter constancy as proposed in Eitrheim and Teräsvirta (1996). Evaluate the dynamic behavior via generalized impulse response function developed by Koop et al. (1996). Modify the model if necessary.

After an adequate model is identified it can be used either for descriptive purposes or for computing forecasts. Forecasting techniques for nonlinear models are studied extensively by Clements and Smith (1997) (see also Granger and Teräsvirta (1993) and Clements and Hendry (1998)).

Specification testing on the first stage, i.e. linearity testing and selecting the transition function, is partially considered in Donauer et al. (2010) by developing a linearity test for the new model. This test will be only briefly reviewed here for the sake of completeness. Starting with the model formulation in (2) with transition function in (5) Donauer et al. (2010) proceed in the spirit of Luukkonen et al. (1988) and approximate the nonlinearity by expanding $\mathcal{G}(\cdot;\gamma,c)$ as a Binomial series which they truncate at a suitable length k. This yields the following general auxiliary regression for a fixed $d \leq p$ and k:

$$y_t = \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{j=1}^{p} \delta_j^{(0)} y_{t-j} + \sum_{j=1}^{p} \delta_j^{(1)} y_{t-j} y_{t-d} + \sum_{j=1}^{p} \delta_j^{(2)} y_{t-j} y_{t-d}^2 + \ldots + \sum_{j=1}^{p} \delta_j^{(2k)} y_{t-j} y_{t-d}^{2k} + u_t \,. \quad (6)$$

The null in this auxiliary model reads:

$$H_0 : \delta_j^{(\ell)} = 0 \quad \text{vs.} \quad H_1 : \text{at least one } \delta_j^{(\ell)} \neq 0; \quad j = 1,\ldots,p \ \text{ and } \ \ell = 0,\ldots,2k \,.$$

This null can be tested by using a standard F-test for a subvector of parameters.

3 Choosing the transition function

Selecting an adequate transition function is crucial to capture the in-sample dynamic of the data generating process properly. A first approach has been made by Teräsvirta and Anderson (1992) and Teräsvirta (1994). These methods concentrate on discriminating between the two most popular STAR models, namely ESTAR and LSTAR. Their proposed technique exploits the properties of the Taylor series expansion of the two nonlinear alternatives and is based on testing a sequence of nested hypotheses. Using such a procedure involves some problems: First the size is not under control anymore because of sequential testing. A second problem is that the classical approach only considers a first order expansion. As Escribano and Jordá (2001) point out this approximation is only adequate if the location parameter c is restricted to zero a priori. To circumvent this and related problems they propose to always include cubic and fourth power terms in the auxiliary regression. This improves the discriminatory power of the test but as this procedure is still based on sequential testing the size is not under control in neither of these approaches.

A rather different approach is proposed by Chen (2003). Based on an encompassing principle for non-nested models from Chen and Kuan (2002) and Chen and Kuan (2007) he directly test whether an ESTAR or LSTAR formulation is more adequate. In simulations he shows that his approach is more powerful in detecting the correct specification.

The T-STAR model was proposed as an alternative to the ESTAR model and therefore we concentrate on distinguishing between T-STAR and LSTAR model formulations.

Our approach is related to Chen (2003) in the sense that we also aim to directly test whether T-STAR or LSTAR is more appropriate. The approach we take is an encompassing approach to conditional mean testing for non-nested hypotheses as proposed by Wooldridge (1990a). The idea is that the residuals of the model under the null estimated by nonlinear least squares should be orthogonal to the gradient of the null model and the residuals should also be independent of the gradient of the model under the alternative. More formally, let

$$\{m_t(w_t,\alpha) : \alpha \in A, t = 1,2,\ldots\}, \quad A \subset \mathbb{R}^P \tag{7}$$

be the model under the null. In our case this is the T-STAR model in (2) together with the transition function in (5). The model under the alternative reads

$$\{\mu_t(w_t,\beta) : \beta \in B, t = 1,2,\ldots\}, \quad B \subset \mathbb{R}^Q . \tag{8}$$

In our case this is an LSTAR model defined as in (2) with the transition function

$$\mathcal{G}(\cdot;\gamma,c) = \{1+\exp(-\gamma(y_{t-d}-c))\}^{-1} . \tag{9}$$

Define the residuals under the null as

$$\hat{e}_t = y_t - \hat{m}_t(w_t, \hat{\alpha}) \; .$$

The test considers to test for $\delta = 0$ in

$$y_t = m_t(w_t, \alpha) + \delta \nabla_\beta \mu(\hat{\beta}) + error_t, \tag{10}$$

where $\nabla_\beta \mu(\hat{\beta}) = \nabla_\beta \mu(w_t, \hat{\beta})$ is the $Q \times 1$ gradient of $\mu_t(w_t, \beta)$ evaluated at the nonlinear least squares estimate $\hat{\beta}$ of β. Denote by $\nabla_\alpha m(\hat{\alpha})$ the same $P \times 1$ quantity for the null model $m_t(w_t, \alpha)$. The test can be carried out by computing

$$LM = TR^2, \tag{11}$$

where T is the sample size and R^2 is the coefficient of determination from the regression of $\hat{e}_t$ on $\nabla_\beta \mu(\hat{\beta})$ and $\nabla_\alpha m(\hat{\alpha})$. Asymptotically this test statistic follows a $\chi^2(Q)$ distribution.
If no homoscedasticity is assumed the test can be robustified by using results from Wooldridge (1990b) and run through the steps described in *Procedure 3.1* in Wooldridge (1990a, p. 336).

4 Residual based misspecification tests

4.1 Test of serial independence

In deriving the evaluation tests for the T-STAR model we roughly follow the steps taken in Eitrheim and Teräsvirta (1996). However, there are some crucial differences between their tests and the ones developed during the course of this section.
Consider the following general T-STAR model of order p

$$y_t = [\Psi w_t] + [\Phi w_t] \times \left[1 - \left\{ 1 + (y_{t-d} - c)^2 \right\}^{-\gamma} \right] + u_t \; , \tag{12}$$

with $\Psi = (\psi_0, \psi_1, \ldots, \psi_p)$, $\Phi = (\varphi_0, \varphi_1, \ldots, \varphi_p)$ and $w_t = (1, y_{t-1}, \ldots, y_{t-p})'$. The innovation process u_t is autocorrelated and follows an invertible moving-average process of order q (MA(q))[2]

$$u_t = [1 + \alpha(L)] \varepsilon_t; \quad \varepsilon_t \overset{iid}{\sim} (0, \sigma^2) \; , \tag{13}$$

[2]This is not restrictive as this is also the test against innovations following an AR(q) process since they are asymptotically local equivalent alternatives of each other (see Godfrey (1988, p. 114)).

where L denotes the lag operator and $\alpha(L) = (\alpha_1 L + \alpha_2 L^2 + \ldots + \alpha_q L^q)$.
Under the null of no serial correlation the innovations u_t in (12) are *iid* and under the alternative they follow (13). The testable pair of hypotheses thus reads

$$H_0 : \alpha_i = 0 \quad \text{vs.} \quad H_1 : \text{at least one } \alpha_i \neq 0; \quad i = 1, \ldots, q \, .$$

Based on results given in Godfrey (1988), we can proceed to develop a Lagrange multiplier (LM) test against serially correlated innovations.
To keep the notation simple define the so-called skeleton (see Franses and van Dijk (2000))

$$f(w_t; \Gamma) := [\Psi w_t] + [\Phi w_t] \times \left[1 - \left\{ 1 + (y_{t-d} - c)^2 \right\}^{-\gamma} \right] , \tag{14}$$

with $\Gamma = (\Psi, \Phi, \gamma, c)$ and the model becomes

$$y_t = f(w_t; \Gamma) + u_t \, .$$

Further define

$$\hat{u}_t := y_t - f(w_t; \hat{\Gamma}) \, ,$$

where $\hat{\Gamma} = (\hat{\Psi}, \hat{\Phi}, \hat{\gamma}, \hat{c})$ is the minimizer of

$$Q = \sum_{t=1}^{T} [y_t - f(w_t; \Gamma)]^2 \, .$$

The test statistic for the null of no serial correlation can now be obtained as

$$LM_{(1)} = TR^2 \, , \tag{15}$$

where T is the sample size and R^2 is the coefficient of determination from the regression of $\hat{u}_t$ on lagged residuals $\hat{u}_{t-1}, \ldots, \hat{u}_{t-q}$ and $\hat{z}_t = \frac{\partial f(w_t;\Gamma)}{\partial \Gamma}\Big|_{\hat{\Gamma}}$. This '$TR^2$' variant of the LM test has been proposed for detecting misspecifications by Breusch (1978) and Godfrey (1978) (see also Godfrey (1988)).
Eitrheim and Teräsvirta (1996) provide the components needed for computing (15) for the

ESTAR and LSTAR case. The respective components for the T-STAR case are as follows:

$$\frac{\partial f(w_t;\Gamma)}{\partial \Psi} = w_t \tag{16}$$

$$\frac{\partial f(w_t;\Gamma)}{\partial \Phi} = w_t \times \left[1-\left\{1+(y_{t-d}-c)^2\right\}^{-\gamma}\right] \tag{17}$$

$$\frac{\partial f(w_t;\Gamma)}{\partial \gamma} = [\Phi\, w_t] \times \left\{\left[1+(y_{t-d}-c)^2\right]^{-\gamma} \ln\left(1+(y_{t-d}-c)^2\right)\right\} \tag{18}$$

$$\frac{\partial f(w_t;\Gamma)}{\partial c} = [\Phi\, w_t] \times \left\{\frac{\left[1+(y_{t-d}-c)^2\right]^{-\gamma} \gamma(-2\gamma+2c)}{1+(y_{t-d}-c)^2}\right\} . \tag{19}$$

Eitrheim and Teräsvirta (1996) point out that such a test can suffer from size distortions in finite samples because the estimation procedure in the first step may result in a solution in which the residuals are not perfectly orthogonal to the gradient $\hat{z}_t$. We adopt their proposed remedy for this situation and take an extra orthogonalization step after estimating the model in (12) and obtaining the residuals. The test can now be performed in three stages:

(i) Estimate the T-STAR model using either nonlinear least squares or conditional (quasi) maximum likelihood under the null and obtain the residuals $\hat{u}_t$.

(ii) Regress $\hat{u}_t$ on the gradient $\hat{z}_t$ and obtain the residuals $\breve{u}_t$.

(iii) Regress $\breve{u}_t$ on $\breve{u}_{t-1}, \ldots, \breve{u}_{t-q}$ and the partial derivatives of $f(w_t;\Gamma)$ evaluated at $\hat{\Gamma}$ as detailed in (16) - (19) and compute the respective R^2.

Under the null of no serial correlation the test statistic follows

$$LM_{(1)} \sim \chi^2(q) .$$

4.2 Test of no remaining nonlinearity

Since there a numerous ways in which a nonlinear model can be misspecified we restrict ourselves to the case of additive nonlinearity. Therefore consider the model

$$y_t = [\Psi\, w_t] + [\Phi\, w_t] \times \mathcal{G}_1(y_{t-d};\gamma_1,c_1) + [\Xi\, w_t] \times \mathcal{G}_2(y_{t-e};\gamma_2,c_2) + \varepsilon_t , \tag{20}$$

where $\mathcal{G}_1(\cdot)$ and $\mathcal{G}_2(\cdot)$ are transition functions of the form in (5). As the null model we consider (20) but without the second nonlinear component. The respective null can be formulated as

$$H_0 : \gamma_2 = 0 \quad \text{vs.} \quad H_1 : \gamma_2 > 0 \, .$$

This situation can also be interpreted as testing a two regime T-STAR model against a three regime T-STAR model. In this interpretation the test is readily extendable to more general models containing more than two or three regimes.[3]

If we want to test the pair of hypotheses in (21) we face a similar problem as Luukkonen et al. (1988) and Donauer et al. (2010) when constructing linearity tests, i.e. that under the null the model in (20) is not fully identified. We circumvent this problem similarly and approximate the second nonlinearity by using an adequate linear series expansion around $\gamma_2 = 0$. As a Taylorian expansion is impractical here we follow Donauer et al. (2010) and use a Binomial series expansion which we truncate after $k = 3$ summands.

The linear approximation to $\mathcal{G}_2(\cdot)$ reads

$$\mathcal{G}_2^{(3)} = \gamma_2(y_{t-e} - c_2)^2 - \frac{1}{2}\gamma_2(\gamma_2+1)(y_{t-e} - c_2)^4 + \frac{1}{6}\gamma_2(\gamma_2+1)(\gamma_2+2)(y_{t-e} - c_2)^6 \, . \tag{21}$$

After substituting the transition function and combining terms we obtain the auxiliary model

$$\begin{aligned} y_t &= [\Psi w_t] + [\Phi w_t] \times \mathcal{G}_1(y_{t-d}; \gamma_1, c_1) + \qquad (22) \\ & \quad \delta_0 w_t + \delta_1 w_t y_{t-e} + \delta_2 w_t y_{t-e}^2 + \delta_3 w_t y_{t-e}^3 + \delta_4 w_t y_{t-e}^4 + \delta_5 w_t y_{t-e}^5 + \delta_6 w_t y_{t-e}^6 + r_t \, , \end{aligned}$$

where $\delta_i, \ i = 0, \ldots, 6,$ are functions of the parameters Ξ, γ_2 and c_2 given in the appendix A.

This reformulation solves the identification problem as the parameters γ_2, c_2 and Ξ are now multiplicatively connected. The innovation term is now denoted by r_t as it not only contains ε_t but also the approximation error from truncating the infinite Binomial series. Notice that under H_0, $r_t = \varepsilon_t$.

The pair of hypotheses for the auxiliary model reads

$$H_0 : \delta_i = 0 \quad \text{vs.} \quad H_1 : \text{at least one } \delta_i \neq 0 \, ; i = 0, \ldots, 6 \, .$$

[3] To test against a very general form of remaining nonlinearity in the residuals, artificial neural network tests as studied by Lee et al. (1993) could be used. See also Teräsvirta et al. (2008).

This most general case simplifies if the location parameter c_2 is set to zero a priori which is frequently done in empirical applications. Then only the odd powers remain in the auxiliary model.

The test can be carried out using the test statistic in (15). The corresponding R^2 is obtained from regressing the residuals obtained under the null, i.e. model (20) without the second nonlinearity, on the partial derivatives of the regression function evaluated under the null, i.e. $\hat{z}_t$ given in (16) - (19) and the auxiliary regressors w_t and $w_t y_{t-e}^i$, $i = 1, \ldots, 6$. After the estimation of the null model the additional orthogonalization step (ii) as for the test against serially correlated innovations can be performed to avoid numerical problems as described at the end of section 4.1.

The resulting test statistic follows

$$LM_{(2)} \sim \chi^2 (7(p+1)) \ .$$

Note that in the model formulation (20) the delay parameter of the second nonlinear component is assumed to be e with $e \neq d$ but $e \leq p$. Similar to determine the delay d described in step 1 of the modeling cycle the test can be carried out for various values of e and the test yielding the minimal p-value is chosen as the decisive test decision.

4.3 Test of parameter constancy

Testing for the constancy of estimated parameters a well established way of checking the adequacy of linear models (see e.g. Chow (1960), Quandt (1960) or Andrews (1993)). In the context of nonlinear time series this maintains its importance but the assumption of an abrupt break in the parameters is questionable. Therefore, we propose a parametric test of the null of parameter constancy against the alternative that the autoregressive parameters change smoothly over time. Assuming the parameters of the transition function fixed the model under the alternative reads

$$y_t = [\Psi(t)\, w_t] + [\Phi(t)\, w_t] \times \mathcal{G}(y_{t-d}; \gamma, c) + \varepsilon_t \ , \qquad (23)$$

with $\varepsilon_t \overset{iid}{\sim} N(0,1)$. The parameter vectors $\Psi(t)$ and $\Phi(t)$ are now functions of time and can be represented as

$$\Psi(t) = \overline{\Psi} + \lambda_1 \mathcal{K}(t;\gamma_1, c_1) \tag{24}$$

$$\Phi(t) = \overline{\Phi} + \lambda_2 \mathcal{K}(t;\gamma_1, c_1)\ , \tag{25}$$

where $\lambda_i,\ i = 1,2$, are vectors conformable to the dimension of Ψ and Φ and $\mathcal{K}(\cdot)$ has the functional form

$$\mathcal{K}(t;\gamma_1, c_1) = 1 - \left\{1 + (t - c_1)^2\right\}^{-\gamma_1};\ \gamma_1 > 0\ . \tag{26}$$

This function induces a nonmonotonic change which is symmetric around $t = c_1$. If $\mathcal{K}(\cdot)$ takes on the limiting case $\gamma_1 \to \infty$ then $\mathcal{K}(\cdot) \to 1 - \mathbb{1}_{c_1}$ which corresponds to a single abrupt break only at $t = c_1$. $\mathbb{1}_{c_1}$ denotes the indicator function at c_1. The null of parameter constancy against the alternative of smoothly changing parameters over time can now be expressed as

$$H_0 : \gamma_1 = 0 \quad \text{vs.} \quad H_1 : \gamma_1 > 0\ .$$

Again we face an identification problem under the null as γ_1 is not identified. We expand $\mathcal{K}(\cdot)$ as Binomial series and truncate after $k = 3$ summands. This yields

$$\mathcal{K}^{(3)} = \gamma_1(t - c_1)^2 - \frac{1}{2}\gamma_1(\gamma_1 + 1)(t - c_1)^4 + \frac{1}{6}\gamma_1(\gamma_1 + 1)(\gamma_1 + 2)(t - c_1)^6\ . \tag{27}$$

Upon substitution of the approximation in (27) into the model in (23) we obtain after combining terms the auxiliary regression

$$\begin{aligned} y_t \quad = \quad & \left[\overline{\Psi} w_t + \delta_0 w_t + \delta_1 t w_t + \delta_2 t^2 w_t + \delta_3 t^3 w_t + \delta_4 t^4 w_t + \delta_5 t^5 w_t + \delta_6 t^6 w_t\right] + \\ & \left[\overline{\Phi} w_t + \beta_0 w_t + \beta_1 t w_t + \beta_2 t^2 w_t + \beta_3 t^3 w_t + \beta_4 t^4 w_t + \beta_5 t^5 w_t + \beta_6 t^6 w_t\right] \times \mathcal{G}(y_{t-d};\gamma, c) + r_t\ , \end{aligned} \tag{28}$$

where under the null $r_t = \varepsilon_t$. The coefficients $\delta_i,\ i = 0,\dots,6,$ and $\beta_i,\ i = 0,\dots,6,$ are functions of γ_1, c_1 and $\lambda_i,\ i = 1,2$ and given in the appendix A.
The pair of hypotheses for the auxiliary model reads

$$H_0 : \delta_i = \beta_i = 0 \quad \text{vs.} \quad H_1 : \text{at least one } \delta_i \text{ or } \beta_i \neq 0\ ; i = 0,\dots,6\ .$$

The test against smoothly changing parameters can now be computed using (15) where the R^2 is obtained from the regression of the residuals under the null on the gradient $\hat{z}_t$ and the

auxiliary regressors. The additional orthogonalization step (ii) is again recommended. The additional regressors in (28) are trending but using Theorem 1 in Lin and Teräsvirta (1994) the OLS estimates are still normally distributed and the usual asymptotic holds.
The test statistic follows

$$LM_{(3)} \sim \chi^2(14(1+p)) \ .$$

As with the test against remaining nonlinearity the auxiliary regression simplifies when $c_1 = 0$ is assumed a priori. Then only the even powers of the trend remain.

4.4 Generalized impulse response function

Impulse response functions (IRF) are a well established way to analyze the effect of a shock on the behavior of a time series model. Traditional impulse response analysis therefore considers the question: 'What is the effect of a shock of size δ hitting the system at time t on the state of the system at time $t+n$, if no other shock hits the system in the meantime?'. Denote with δ the size of a shock hitting the system at time t and with ω_{t-1} a particular realization of the information set Ω_{t-1} then we can define the impulse response function more formally as

$$\begin{aligned} IRF(n,\delta,\omega_{t-1}) &= E[y_{t+n}|\varepsilon_t = \delta, \varepsilon_{t+1} = \ldots = \varepsilon_{t+n} = 0, \omega_{t-1}] - \\ & \quad E[y_{t+n}|\varepsilon_t = \varepsilon_{t+1} = \ldots = \varepsilon_{t+n} = 0, \omega_{t-1}] \ . \end{aligned} \tag{29}$$

The second conditional expectation is often called the 'baseline' which acts as a reference point. For linear models Koop et al. (1996) point out three properties of the impulse response functions: *Symmetry*, i.e. a shock of -1 has exactly the opposite effect of a shock of +1, *shock linearity*, i.e. a shock of size 2 has exactly twice the effect as a shock of 1, and the IRF is *history independent*, i.e. the past does not effect the response in any way. The authors also provide various examples to show that these properties do not carry over to the nonlinear case. To remedy this drawbacks Koop et al. (1996) propose a generalized impulse response function (GIRF) which is itself a random variable and is defined as

$$GIRF(n,\delta,\omega_{t-1}) = E[y_{t+n}|\varepsilon_t = \delta, \omega_{t-1}] - E[y_{t+n}|\omega_{t-1}] \ . \tag{30}$$

Here, the conditional expectation is conditioned only on the shock δ and the past ω_{t-1}. The shocks occurring in the meantime are handled by averaging them out. For computing the

GIRF obviously we need the conditional expectation of a nonlinear model which is cumbersome as the dimension of the integral defining the conditional expectation grows with n (see Granger and Teräsvirta (1993)). To ease implementation Koop et al. (1996) propose a numerical technique to compute the conditional expectation by means of Monte Carlo integration (for further details see Koop et al. (1996, p. 135)).

The history on which we condition the GIRF can also be only a subset of the entire history such as $\omega_{t-1} \in \mathcal{A}$. Where $\mathcal{A}$ could be the subset containing only the observations coming from one regime. Such an approach is useful in determining whether the dynamic behavior is different in periods of recession compared with expansionary periods (see e.g. van Dijk et al. (2002a) and Kapetanios (2003)).

The GIRF can also be used to analyze whether the model under consideration produces asymmetric effects over time. This could be done as in Potter (1995) by defining

$$ASYM(n, \delta, \omega_{t-1}) = GIRF(n, \delta, \omega_{t-1}) + GIRF(n, -\delta, \omega_{t-1}) \,. \tag{31}$$

Another use of the GIRF is to examine the persistence of shocks (see Koop et al. (1996)). If a time series model is stationary, at least globally, then the effect of a shock should eventually fade away to zero if the horizon n goes to infinity. As a consequence the density of the GIRF defined by (30) should collapse to a single spike at zero. Therefore, the dispersion of the densities of the GIRF at different horizons n can be used as a pragmatic measure of the persistence of shocks.

5 Finite sample properties

To study the behavior of the tests in finite samples we conduct a small scale simulation study. We report size results from simulating the following T-STAR process

$$y_t = 0.7 y_{t-1} - 0.5 y_{t-1} \left[1 - \left\{ 1 + y_{t-1}^2 \right\}^{-1} \right] + \varepsilon_t \,, \tag{32}$$

where $\varepsilon_t \overset{iid}{\sim} N(0,1)$. The location parameter is set to $c = 0$ only to reduce computational burden in the estimation process and it does not effect the results reported here.

We study different sample sizes of $T = 300, 500, 1000$. For all time series generated we discard the first 500 observations in order to be independent of the initial values. The first step in each simulation is to compute the linearity test against T-STAR proposed by Donauer et al. (2010). If the null cannot be rejected at the 5% level of significance the series is discarded and a new one is simulated. If the null is rejected the size or power experiment is conducted. This is done until the number of replication $M = 50000$ is reached. Applying the linearity test in the first step is done to avoid the estimation of a series in which there is not much evidence of nonlinearity.

The results of the size experiment for the test against LSTAR models from section 3 are shown in Table 1.

α	$T = 300$	$T = 500$	$T = 1000$
1%	0.950	0.912	0.922
5%	4.822	4.844	4.750
10%	9.900	9.890	9.956

Table 1: Empirical size of the test against LSTAR [in %].

The size of the test procedure to choose between competing STAR formulations shows virtually no distortions in the considered sample sizes. This is especially notable as the encompassing test to discriminate between ESTAR and LSTAR proposed by Chen (2003) is generally undersized.

The power of the test against LSTAR was simulated using the following LSTAR specification as alternative:

$$y_t = 0.7y_{t-1} - 0.5y_{t-1}\,[1 + \exp(-2y_{t-1})]^{-1} + \varepsilon_t\ .$$

The results are given in Table 2.

α	$T = 300$	$T = 500$	$T = 1000$
1%	21.050	39.078	67.472
5%	39.838	57.910	78.750
10%	50.675	66.920	83.512

Table 2: Empirical power of the test against LSTAR [in %].

The power of the testing procedure shows reasonable discriminatory power of the test. In particular it yields better results than the selection procedures of Teräsvirta and Anderson (1992) and Teräsvirta (1994), relying on the simulation results in Chen (2003). Compared to the test procedure for the ESTAR–LSTAR case of the latter author the power is comparable in most settings. In some cases the power of the test of Chen (2003) is clearly higher but given the serious size distortions[4] the power of his test is not readily interpretable.

When studying the empirical power of the test against serially correlated innovations described in section 4.1 we simulate from (32) but assume that the innovation process follows an AR(1) process $u_t = \rho u_{t-1} + \varepsilon_t$, with $\varepsilon_t \overset{iid}{\sim} N(0,1)$ and $\rho = 0.2, 0.4, 0.6$.

The results for the size and power experiment are summarized in Table 3 and Table 4 respectively.

	$T = 300$			$T = 500$			$T = 1000$		
α	$q = 1$	$q = 2$	$q = 5$	$q = 1$	$q = 2$	$q = 5$	$q = 1$	$q = 2$	$q = 5$
1%	1.066	1.044	0.990	0.982	1.072	0.910	1.074	0.982	1.014
5%	5.254	5.208	5.022	5.074	5.186	4.892	5.362	5.028	5.100
10%	10.302	10.506	10.104	10.234	10.438	10.032	10.164	10.240	10.100

Table 3: Empirical size of the test of no innovation correlation [in %].

The results in Table 3 show that the empirical size is always very close to its nominal level. Although some minor distortions are visible the overall result confirms a satisfactorily behavior of the test in finite samples.

[4]In some settings he obtains a size of only 0.5% at a nominal $\alpha = 5\%$ level.

$\rho = 0.2$	$T = 300$			$T = 500$			$T = 1000$		
α	$q = 1$	$q = 2$	$q = 5$	$q = 1$	$q = 2$	$q = 5$	$q = 1$	$q = 2$	$q = 5$
1%	4.030	7.210	4.112	18.004	12.424	7.038	38.910	29.094	17.606
5%	10.036	20.176	13.880	38.422	30.090	20.644	62.968	52.822	37.888
10%	14.276	30.718	23.064	50.936	42.136	31.470	74.304	65.226	50.902
$\rho = 0.4$	$T = 300$			$T = 500$			$T = 1000$		
α	$q = 1$	$q = 2$	$q = 5$	$q = 1$	$q = 2$	$q = 5$	$q = 1$	$q = 2$	$q = 5$
1%	43.138	32.416	19.384	68.482	57.274	40.060	95.498	91.406	81.562
5%	67.250	56.998	41.046	86.474	78.716	64.146	98.962	97.478	93.382
10%	77.660	68.994	54.452	92.132	86.608	75.366	99.568	98.828	96.610
$\rho = 0.6$	$T = 300$			$T = 500$			$T = 1000$		
α	$q = 1$	$q = 2$	$q = 5$	$q = 1$	$q = 2$	$q = 5$	$q = 1$	$q = 2$	$q = 5$
1%	75.702	65.084	46.986	94.442	89.720	77.784	99.950	99.814	99.100
5%	90.392	84.140	70.484	98.626	97.014	91.434	99.994	99.986	99.846
10%	94.778	90.658	80.636	99.398	98.576	95.508	99.996	99.992	99.946

Table 4: Empirical power of the test of no innovation correlation [in %].

The results for the empirical power displayed in Table 4 show a similar behavior to the test for the ESTAR case described in Eitrheim and Teräsvirta (1996). The power slightly decreases if the tested order of autocorrelation q increases. This might be expected as $q = 1$ is the true data generating process. Another factor that influences the power is the degree of autocorrelation ρ. In finite samples and a low degree of serial correlation the power is quite low but increases steeply if the sample size and/or the ρ becomes larger. Such a behavior is somewhat expected as the χ^2 distribution holds only asymptotically and if ρ increases the serial correlation becomes easier to detect. In general the test yields good results for most situations encountered in practice and helps to reveal severe misspecifications.

For the assessment of the test of no remaining nonlinearity from section 4.2 we simulate data from (32) to perform the size experiment. The results for the size and power experiments are displayed in Tables 5 and 6 respectively.

α	$T = 300$	$T = 500$	$T = 1000$
1%	0.806	0.834	0.852
5%	4.552	4.728	4.640
10%	9.540	9.600	9.566

Table 5: Empirical size of the test of no remaining nonlinearity [in %].

The results of the size experiment show that the empirical size is close to its nominal level. If anything, the test is slightly conservative.

For the power experiment we simulate data from

$$y_t = 0.3y_{t-1} - 0.1y_{t-1}\left[1 - \left\{1 + y_{t-1}^2\right\}^{-1}\right] + 0.75y_{t-1}\left[1 - \left\{1 + y_{t-1}^2\right\}^{-3.5}\right] + \varepsilon_t\ ,$$

where $\varepsilon_t \overset{iid}{\sim} N(0,1)$. Additionally to this data generating process we simulate from an LSTAR(1) and ESTAR(1) process and induce remaining nonlinearity by fitting the wrong model, namely an T-STAR(1). The respective processes read

$$\begin{aligned} y_t &= 0.7y_{t-1} - 0.5y_{t-1}\left[1 + \exp(-2y_{t-1})\right]^{-1} + \varepsilon_t \\ y_t &= 0.7y_{t-1} - 0.5y_{t-1}\left[1 - \exp(-2y_{t-1}^2)\right] + \varepsilon_t\ , \end{aligned}$$

with $\varepsilon_t \overset{iid}{\sim} N(0,1)$. The results for these simulations are presented in Table 6.

Another variant of the power simulation inspired by Eitrheim and Teräsvirta (1996) is also explored: The data is generated by (32) but misspecification is assumed by fitting a STAR model of the wrong kind to the data, namely a LSTAR(1). These results are presented in Table 7.

H_1 : T-STAR			
α	$T = 300$	$T = 500$	$T = 1000$
1%	3.120	2.986	3.772
5%	12.224	11.424	12.184
10%	21.474	20.076	20.676
H_1 : LSTAR			
α	$T = 300$	$T = 500$	$T = 1000$
1%	4.430	5.522	8.274
5%	13.924	16.898	27.044
10%	24.070	28.614	43.822
H_1 : ESTAR			
α	$T = 300$	$T = 500$	$T = 1000$
1%	1.348	1.292	1.276
5%	6.848	6.480	6.210
10%	13.440	12.716	11.990

Table 6: Empirical power of the test of no remaining nonlinearity [in %].

For this test the empirical power results reveal a comparable performance for this test as for the test in the ESTAR case reported in Eitrheim and Teräsvirta (1996). The test appears to have reasonably good power against LSTAR models especially in larger samples where the power increases quite steeply.

Interestingly the test appears to have some nontrivial power also against ESTAR at least in small samples. This is surprising given that T-STAR has been designed to resemble the desirable properties of ESTAR. This power vanishes as T increases underlining that the T-STAR model can very well serve as an alternative to ESTAR as they can hardly be distinguished for reasonable sample sizes.

α	$T = 300$	$T = 500$	$T = 1000$
1%	31.512	34.200	41.434
5%	49.670	52.924	60.868
10%	61.682	64.914	71.762

Remark: An LSTAR(1) model was fitted to data generated from (32).

Table 7: Empirical power of the test of no remaining nonlinearity [in %].

Analyzing the power experiment set up as in Eitrheim and Teräsvirta (1996) we obtain rather good results for the test against no remaining nonlinearity even in finite samples. In particular we obtain higher power as in the ESTAR case.

Turning to the results for the test of parameter constancy we report size and power results in Tables 8 and 9 respectively.

α	$T = 300$	$T = 500$	$T = 1000$
1%	0.778	0.850	0.940
5%	4.490	4.504	4.784
10%	9.278	9.354	9.522

Table 8: Empirical size of the test of parameter constancy [in %].

The test shows only minor size distortions in finite samples and approaches its nominal level as the sample size increases. Overall the test seems to be conservative, if anything.

For the power simulations we generate data from (23) of order one and set $\psi_0(t) = 2\mathcal{K}(\cdot)$, $\psi_1(t) = -0.2$, $\vartheta_0(t) = 0$ and $\vartheta_1(t) = (1.1 - 0.9\mathcal{K}(\cdot))$ where $\mathcal{K}(\cdot) = \mathcal{K}(t/T; 3, 0)$ as in (26).

α	$T = 300$	$T = 500$	$T = 1000$
1%	3.008	22.110	99.248
5%	11.458	44.252	99.746
10%	19.528	57.336	99.824

Table 9: Empirical power of the test of parameter constancy [in %].

The test shows reasonable power to detect parameter changes in finite samples. If the sample size increases the power of the test increases very steeply. Thus the test is a useful tool to detect parameter changes in most sample sizes.

6 Modeling real exchange rates

To demonstrate the application of the test developed in this paper we run through the whole modeling cycle described in section 2 to model real exchange rates.
We use the same data that has been analyzed by Taylor et al. (2001) and by Rapach and Wohar (2006). Namely, we analyze monthly real exchange data for Germany against the US from 1980:01 - 1994:12 ($T = 288$).[5] The series is depicted in Figure 1.

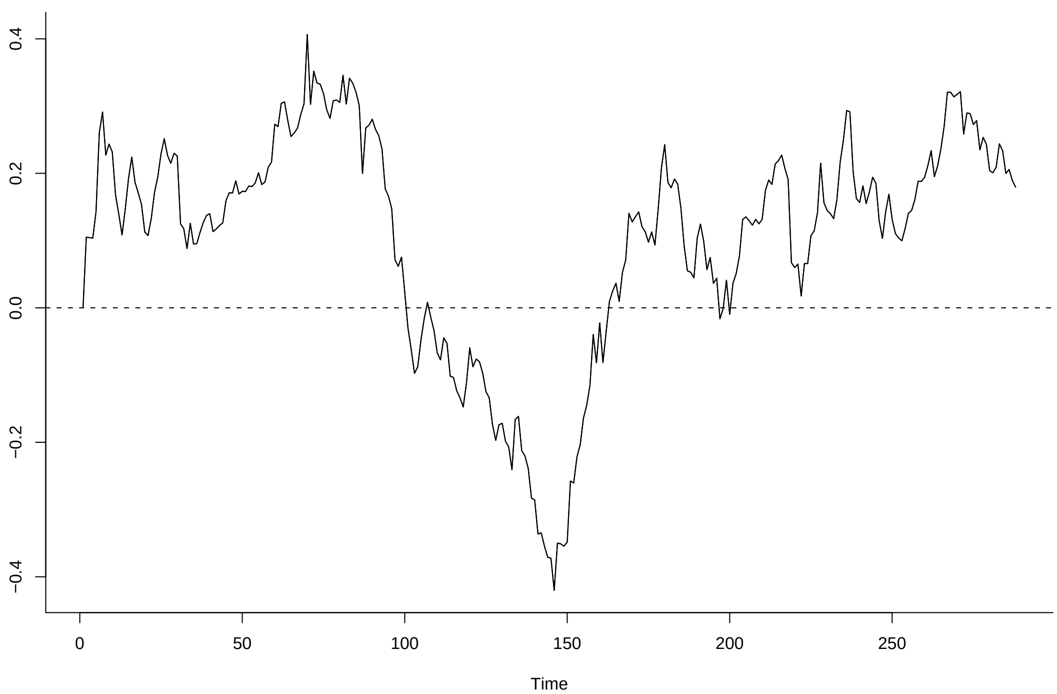

Figure 1: Monthly log real exchange rate for Germany.

Determining the lag length using the consistent BIC we obtain $p = 1$. The linearity test rejects the null of linearity on the $\alpha = 5\%$ level of significance. The test against LSTAR yields a test decision in favor of the null model, i.e. T-STAR.

[5] The data set is available from David Rapach's website at: http://pages.slu.edu/faculty/rapachde/Nlfit.zip.

As the data set has also been analyzed by Taylor et al. (2001) and by Rapach and Wohar (2006) we report also their estimates for an ESTAR model. The estimated model has been theoretically justified by the assumption that real exchange rates follow a nonlinear STAR model with one unit root regime and one stationary regime that pulls the real exchange rate back into its stable equilibrium once it wanders too far off. The model reads

$$y_t = y_{t-1} + \pi y_{t-1} \mathcal{G}(\cdot) + \varepsilon_t \; , \tag{33}$$

where $-2 < \pi < 0$ to ensure global stationarity of the model. The transition function $\mathcal{G}(\cdot)$ is either as in (3) for the ESTAR model or as in (5) for the T-STAR model. Additionally Taylor et al. (2001) and Rapach and Wohar (2006) set $\pi := -1$.

The estimation results are in table 10.

	ESTAR	T-STAR
$\hat{\pi}$	-1	-0.023
$\hat{\gamma}$	0.264	275.284
$\hat{\sigma}_{\varepsilon}$	0.035	0.032

Table 10: Estimation of STAR models.

Albeit the estimates for the T-STAR model might look puzzling at first Donauer et al. (2010) show that these estimates are much more reasonable than the corresponding ESTAR estimates as the ESTAR model actually degenerates to a random walk as opposed to the T-STAR model which maintains the regime switching behavior. Further support of the PPP can be seen in Figure 2.

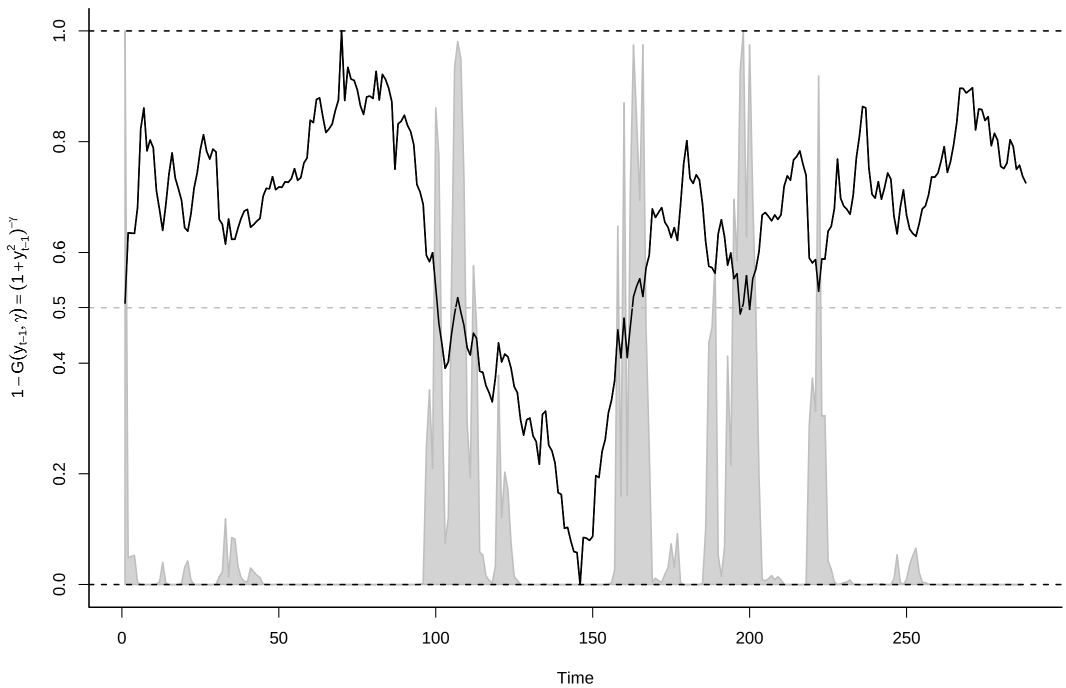

Figure 2: Monthly log real exchange rate for Germany and transition function.

The figure shows the time series, rescaled to be in the closed interval $[0,1]$, and one minus the estimated transition function over time. The gray shaded area are the time periods in which the process behaves like a random walk. This is always the case when the process is close to its equilibrium near zero (note that the dotted line at 0.5 is the zero line of the unscaled series). Thus we have a stationary but nonlinear process most of the time which behaves like a random walk near the equilibrium as predicted by the PPP. Donauer et al. (2010) further show that the data is globally stationary although one unit root regime is present. Global stationarity is important for the asymptotic distributions of the misspecification tests to hold. Performing the misspecification tests yields the results in table 11.

	Test Statistic	Critical Value
Test against LSTAR	$LM = 5.779$	$\chi^2_{0.99;\nu=2} = 9.210$
Test of serial independence	$LM_{(1)} = 0.501$	$\chi^2_{0.99;\nu=1} = 6.635$
Test of no remaining nonlinearity	$LM_{(2)} = 8.935$	$\chi^2_{0.99;\nu=3} = 11.345$
Test of parameter constancy	$LM_{(3)} = 1.283$	$\chi^2_{0.99;\nu=6} = 16.812$

Table 11: Results of the misspecification tests.

The respective null hypotheses of the tests cannot be rejected at the $\alpha = 0.01$ level of significance hinting at a well specified model.

To gain further insights about the dynamic properties of the fitted model we estimate generalized impulse response functions. As the test against ARCH effects as described in Engle (1982) provides no evidence of conditional heteroscedasticity we randomly sample the innovations with replacement from the estimated model. The shocks we use are $\delta_t = \delta\hat{\sigma}_\varepsilon$, with $\delta = \pm 2, \pm 1$. We compute the GIRF for a horizon of $n = 150$ and estimate the conditional expectations in (30) as means over 5000 Monte Carlo repetitions. Figure 3 shows the estimated impulse response functions.

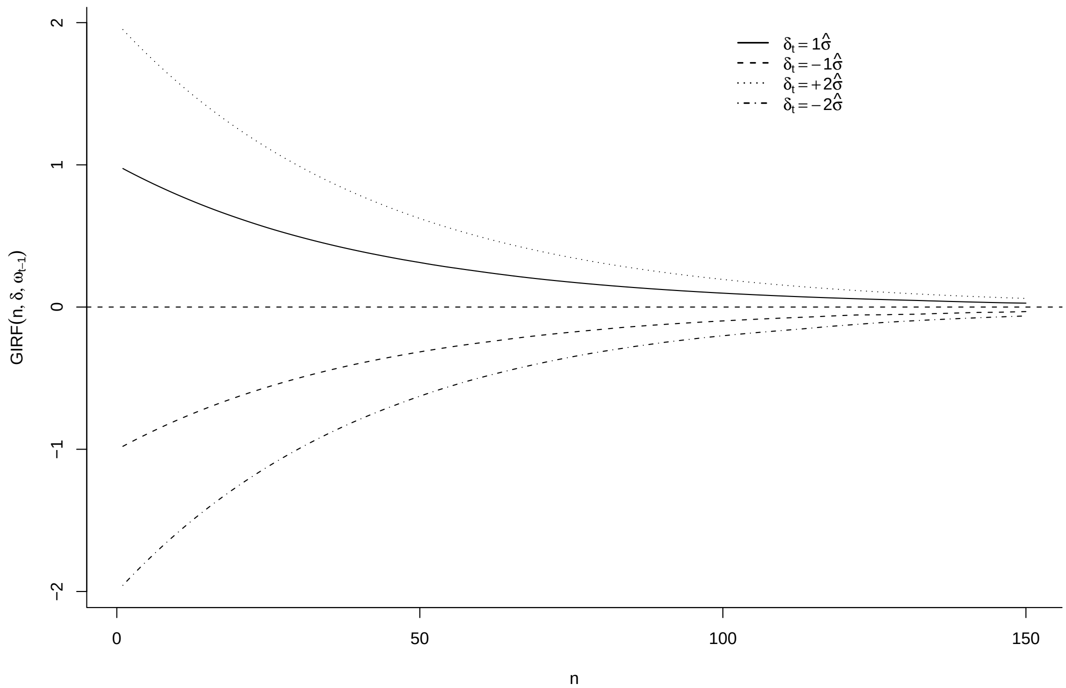

Figure 3: Generalized impulse response functions.

Obviously the shocks hitting the system are very persistent over time but eventually vanish. This supports the parameter estimates that show a highly persistent model. Additionally a high persistence of shocks in the model can also be induced by nonlinearities which in turn leads to such highly persistent impulse response functions (see e.g. van Dijk et al. (2002a) and Kuswanto and Sibbertsen (2008)). Another interesting aspect is whether the response to shocks is asymmetric depending on the sign of the shock. A measure for asymmetry is defined in (31). Figure 4 shows the estimated quantities.

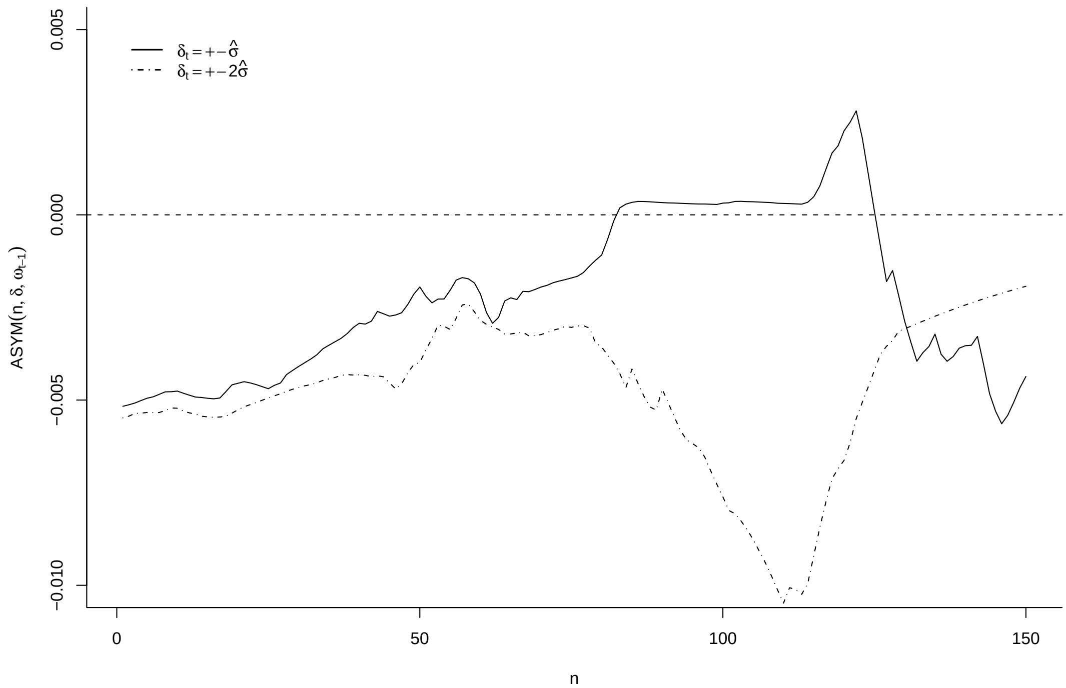

Figure 4: Measure of asymmetry.

For both shocks the behavior is asymmetric depending on the sign of the shock. If the shock is negative but relatively small the response to it is larger compared to a positive shock of the same size. This difference vanishes quite fast as the horizon increases. If however the shock is negative and relatively large ($\delta_t = |2\hat{\sigma}_\varepsilon|$) the response to a negative shock is heavier and decreases much slower. Indeed the asymmetry first increases before it decreases.

7 Conclusion

In this paper we extend the treatment of the newly developed nonlinear time series model named T-STAR developed by Donauer et al. (2010). We consider the modeling cycle for non-linear time series models and contribute to the evaluation stage by proposing LM tests against serially correlated innovations, no remaining nonlinearity and parameter constancy. We also consider evaluation by generalized impulse response functions as proposed by Koop et al. (1996). In simulations we show that all the tests have reasonable power against their respective alternatives and are therefore an useful addition to the evaluation toolbox for nonlinear

T-STAR models.

In an empirical application to real exchange data we put the evaluation techniques to the test and verify that a proposed T-STAR formulation adequately captures the nonlinear behavior of the data. Impulse response analysis is used to further evaluate the dynamic propagation behavior of the estimated model.

A Appendix

A.1 Regression coefficients in (22)

Expanding the expressions containing c_2 as

$$
\begin{aligned}
(y_{t-e}-c_2)^2 &= y_{t-e}^2 - 2y_{t-e}c_2 + c_2^2,\\
(y_{t-e}-c_2)^4 &= y_{t-e}^4 - 4y_{t-e}^3c_2 + 6y_{t-e}^2c_2^2 - 4y_{t-e}c_2^3 + c_2^4,\\
(y_{t-e}-c_2)^6 &= y_{t-e}^6 - 6y_{t-e}^5c_2 + 15y_{t-e}^4c_2^2 - 20y_{t-e}^3c_2^3 + 15y_{t-e}^2c_2^4 - 6y_{t-e}c_2^5 + c_2^6,
\end{aligned}
$$

we obtain after some algebra

$$
\begin{aligned}
\delta_0 &= \left[\left(\gamma_2c_2 - \frac{1}{2}\gamma_2(\gamma_2+1) + \frac{1}{6}\gamma_2(\gamma_2+1)(\gamma_2+2)c_2^6\right)\Xi\right]\\
\delta_1 &= \left[\left(-2\gamma_2c_2 + 2\gamma_2(\gamma_2+1)c_2^3 - \gamma_2(\gamma_2+1)(\gamma_2+2)c_2^5\right)\Xi\right]\\
\delta_2 &= \left[\left(-3\gamma_2(\gamma_2+1)c_2^2 + 2\frac{1}{2}\gamma_2(\gamma_2+1)(\gamma_2+2)c_2^4\right)\Xi\right]\\
\delta_3 &= \left[\left(2\gamma_2(\gamma_2+1)c_2 - 3\frac{1}{3}\gamma_2(\gamma_2+1)(\gamma_2+2)c_2^3\right)\Xi\right]\\
\delta_4 &= \left[\left(-\frac{1}{2}\gamma_2(\gamma_2+1) + 2\frac{1}{2}\gamma_2(\gamma_2+1)(\gamma_2+2)c_2^2\right)\Xi\right]\\
\delta_5 &= \left[(-\gamma_2(\gamma_2+1)(\gamma_2+2)c_2)\Xi\right]\\
\delta_6 &= \left[\left(\frac{1}{6}\gamma_2(\gamma_2+1)(\gamma_2+2)\right)\Xi\right].
\end{aligned}
$$

A.2 Regression coefficients in (28)

Expanding the expressions containing c_1 as

$$
\begin{aligned}
(t-c_1)^2 &= t^2 - 2tc_1 + c_1^2,\\
(t-c_1)^4 &= t^4 - 4t^3c_1 + 6t^2c_1^2 - 4tc_1^3 + c_1^4,\\
(t-c_1)^6 &= t^6 - 6t^5c_1 + 15t^4c_1^2 - 20t^3c_1^3 + 15t^2c_1^4 - 6tc_1^5 + c_1^6\,.
\end{aligned}
$$

we obtain after some algebra for λ_1

$$
\begin{aligned}
\delta_0 &= \left[\left(\gamma_1 c_1 - \frac{1}{2}\gamma_1(\gamma_1+1) + \frac{1}{6}\gamma_1(\gamma_1+1)(\gamma_1+2)c_1^6\right)\lambda_1\right] \\
\delta_1 &= \left[\left(-2\gamma_1 c_1 + 2\gamma_1(\gamma_1+1)c_1^3 - \gamma_1(\gamma_1+1)(\gamma_1+2)c_1^5\right)\lambda_1\right] \\
\delta_2 &= \left[\left(-3\gamma_1(\gamma_1+1)c_1^2 + 2\frac{1}{2}\gamma_1(\gamma_1+1)(\gamma_1+2)c_1^4\right)\lambda_1\right] \\
\delta_3 &= \left[\left(2\gamma_1(\gamma_1+1)c_1 - 3\frac{1}{3}\gamma_1(\gamma_1+1)(\gamma_1+2)c_1^3\right)\lambda_1\right] \\
\delta_4 &= \left[\left(-\frac{1}{2}\gamma_1(\gamma_1+1) + 2\frac{1}{2}\gamma_1(\gamma_1+1)(\gamma_1+2)c_1^2\right)\lambda_1\right] \\
\delta_5 &= \left[\left(-\gamma_1(\gamma_1+1)(\gamma_1+2)c_1\right)\lambda_1\right] \\
\delta_6 &= \left[\left(\frac{1}{6}\gamma_1(\gamma_1+1)(\gamma_1+2)\right)\lambda_1\right]
\end{aligned}
$$

and for λ_2 respectively

$$
\begin{aligned}
\beta_0 &= \left[\left(\gamma_1 c_1 - \frac{1}{2}\gamma_1(\gamma_1+1) + \frac{1}{6}\gamma_1(\gamma_1+1)(\gamma_1+2)c_1^6\right)\lambda_2\right] \\
\beta_1 &= \left[\left(-2\gamma_1 c_1 + 2\gamma_1(\gamma_1+1)c_1^3 - \gamma_1(\gamma_1+1)(\gamma_1+2)c_1^5\right)\lambda_2\right] \\
\beta_2 &= \left[\left(-3\gamma_1(\gamma_1+1)c_1^2 + 2\frac{1}{2}\gamma_1(\gamma_1+1)(\gamma_1+2)c_1^4\right)\lambda_2\right] \\
\beta_3 &= \left[\left(2\gamma_1(\gamma_1+1)c_1 - 3\frac{1}{3}\gamma_1(\gamma_1+1)(\gamma_1+2)c_1^3\right)\lambda_2\right] \\
\beta_4 &= \left[\left(-\frac{1}{2}\gamma_1(\gamma_1+1) + 2\frac{1}{2}\gamma_1(\gamma_1+1)(\gamma_1+2)c_1^2\right)\lambda_2\right] \\
\beta_5 &= \left[\left(-\gamma_1(\gamma_1+1)(\gamma_1+2)c_1\right)\lambda_2\right] \\
\beta_6 &= \left[\left(\frac{1}{6}\gamma_1(\gamma_1+1)(\gamma_1+2)\right)\lambda_2\right].
\end{aligned}
$$

References

Akaike, H. (1974). A new look at the statistical model identification. *IEEE Transactions in Automatic Control*, 19:716–723.

Andrews, D. W. K. (1993). Tests for parameter instability and structural change with unknown change point. *Econometrica*, 61:821–856.

Bacon, D. W. and Watts, D. G. (1971). Estimating the transition between two intersecting straight lines. *Biometrika*, 58:525–534.

Breusch, T. S. (1978). Testing for autocorrelation in dynamic linear models. *Australian Economic Papers*, 17:334–355.

Chen, Y. T. (2003). Discriminating between competing STAR models. *Economics Letters*, 79:161–167.

Chen, Y. T. and Kuan, C. M. (2002). The peusdo-true score encompassing test for non-nested hypotheses. *Journal of Econometrics*, 106:271–295.

Chen, Y. T. and Kuan, C. M. (2007). Corrigendum to ”The peusdo-true score encompassing test for non-nested hypotheses”. *Journal of Econometrics*, 141:1412–1417.

Chow, G. C. (1960). Tests for equality between sets of coefficients in two linear regressions. *Econometrica*, 28:591–605.

Clements, M. P. and Hendry, D. F. (1998). *Forecasting economic time series*. Cambridge University Press.

Clements, M. P. and Smith, J. (1997). The performance of alternative forecasting methods for setar models. *International Journal of Forecasting*, 13:463–475.

Donauer, S., Heinen, F., and Sibbertsen, P. (2010). Identification problems in ESTAR models and a new model. Discussion Paper Leibniz University of Hannover.

Eitrheim, Ø. and Teräsvirta, T. (1996). Testing the adequacy of smooth transition autoregressive models. *Journal of Econometrics*, 74:59–75.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica*, 50:987–1007.

Escribano, A. and Jordá, O. (2001). Testing nonlinearity: Decision rules for selecting between logistic and exponential star models. *Spanish Economic Review*, 3:193–209.

Franses, P. H. and van Dijk, D. (2000). *Non-linear time series models in empirical finance*. Cambridge University Press.

Gatti, D. D., Gallegati, M., and Mignacca, D. (1998). Nonlinear dynamics and european GNP data. *Studies in Nonlinear Dynamics and Econometrics*, 3:43–59.

Godfrey, L. G. (1978). Testing for higher order serial correlation in regression equations when the regressors include lagged dependent variabeles. *Econometrica*, 46:1303–1310.

Godfrey, L. G. (1988). *Misspecification tests in econometrics - The lagrange multiplier principle and other approaches*. Cambridge University Press.

Granger, C. W. J. and Teräsvirta, T. (1993). *Modelling nonlinear economic relationships*. Oxford University Press.

Haggan, V. and Ozaki, T. (1981). Modelling nonlinear random vibrations using an amplitude-dependent autoregressive time series model. *Biometrika*, 68:189–196.

Kapetanios, G. (2003). Threshold models for trended time series. *Empirical Economics*, 28:687–707.

Kapetanios, G., Shin, Y., and Snell, A. (2003). Testing for a unit root in the nonlinear STAR framework. *Journal of Econometrics*, 112:359–379.

Klimko, L. A. and Nelson, P. I. (1978). On conditional least squares estimation for stochastic processes. *The Annals of Statistics*, 6:629–642.

Koop, G., Pesaran, M. H., and Potter, S. M. (1996). Impulse response analysis in nonlinear multivariate models. *Journal of Econometrics*, 74:119–147.

Kuswanto, H. and Sibbertsen, P. (2008). A study on ”Spurious long memory in nonlinear time series models”. Discussion Paper Leibniz University of Hannover No. 410.

Lee, T. H., White, H., and Granger, C. W. J. (1993). Testing for neglected nonlinearity in time series models. *Journal of Econometrics*, 56:269–290.

Lin, C. F. J. and Teräsvirta, T. (1994). Testing the constancy of regression parameters against continuous structural change. *Journal of Econometrics*, 62:211–228.

Luukkonen, R., Saikkonen, P., and Teräsvirta, T. (1988). Testing linearity against smooth transition autoregressive models. *Biometrika*, 75:491–499.

Öcal, N. (2000). Nonlinear models for U.K. macroeconomic time series. *Studies in Nonlinear Dynamics and Econometrics*, 4:123–135.

Potter, S. M. (1995). A nonlinear approach to US GNP. *Journal of Applied Econometrics*, 10:109–125.

Potter, S. M. (1999). Nonlinear time series modelling: An introduction. *Journal of Economic Surveys*, 13:505–528.

Quandt, R. (1960). Tests of the hypotheses that a linear regression system obeys two seperate regimes. *Journal of the American Statistical Association*, 55:324–330.

Rapach, D. E. and Wohar, M. E. (2006). The out-of-sample forecasting performance on nonlinear models of real exchange rate behavior. *International Journal of Forecasting*, 22:341–261.

Rose, A. K. (1988). Is the real interest rate stable? *Journal of Finance*, 43:1095–1112.

Schwarz, G. (1978). Estimating the dimension of a model. *The Annals of Statistics*, 4:461–464.

Taylor, M. P., Peel, D. A., and Sarno, L. (2001). Nonlinear mean-reversion in real exchange rates: Toward a solution to the purchasing power parity puzzles. *International Economic Review*, 42:1015–1042.

Teräsvirta, T. (1994). Specification, estimation, and evaluation of smooth transition autoregressive models. *Journal of the American Statistical Association*, 89:208–218.

Teräsvirta, T. and Anderson, H. M. (1992). Characterizing nonlinearities in business cycles using smooth transition autoregressive models. *Journal of Applied Econometrics*, 7:119–136.

Teräsvirta, T., Lin, C. F., and Granger, C. W. J. (2008). Power of the neural network linearity test. *Journal of Time Series Analysis*, 14:209–220.

Tjøstheim, D. (1986). Estimation in nonlinear time series models. *Stochastic Processes and their Applications*, 21:251–273.

Tong, H. (1990). *Non-linear time series - A dynamical system approach*. Oxford University Press.

Tsay, R. (1986). Non-linearity tests for time series. *Biometrika*, 73:461–466.

van Dijk, D., Franses, P. H., and Paap, R. (2002a). A nonlinear long memory model, with an application to US unemployment. *Journal of Econometrics*, 110:135–165.

van Dijk, D., Teräsvirta, T., and Franses, P. H. (2002b). Smooth transition autoregressive models - a survey of recent developments. *Econometric Reviews*, 21:1–47.

Wooldridge, J. M. (1990a). An encompassing approach to conditional mean tests with applications to testing nonnested hypotheses. *Journal of Econometrics*, 45:331–350.

Wooldridge, J. M. (1990b). A unified approach to robust, regression-based specification tests. *Econometric Theory*, 6:17–43.