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## About the Impact of Model Risk on Capital Reserves: A Quantitative Analysis.

Philip Bertram<sup>†1</sup>, Philipp Sibbertsen<sup>‡</sup> and Gerhard Stahl<sup>§</sup>

<sup>‡</sup>Leibniz University of Hanover, Department of Economics, Germany

<sup>§</sup>Talanx AG, Hanover and Ulm University

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### Abstract

This paper analyzes and quantifies the idea of model risk in the environment of internal model building. We define various types of model risk including estimation risk, model risk in distribution and model risk in functional form. By the quantification of these concepts we analyze the impact of the modeling process of an econometric model on the resulting company model. Utilizing real insurance data we specify, estimate and simulate various linear and nonlinear time series models for the inflation rate and examine its impact on pension liabilities under the aspect of model risk. Under consideration of different risk measures it is shown that model risk can differ profoundly due to the specification process of the econometric model resulting in remarkable monetary differences concerning capital reserves. We furthermore propose a specification strategy for univariate time series models and demonstrate that thereby market risk and capital reserves can be reduced distinctively.

JEL codes: G12, G18

Keywords: Model risk, Estimation risk, Misspecification risk, Basel multiplication factor, Empirical model specification, Capital reserves.

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<sup>1</sup>corresponding author:  
phone: +49- 511- 762-3082, fax: +49- 511- 762-3923  
e-mail: bertram@statistik.uni-hannover.de

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# 1 Introduction

From a financial institution's point of view the importance of dealing with model risk has risen substantially since the implementation of new regulatory laws such as Basel II or Solvency II. Since then the option of implementing internal models instead of the hitherto obligatory application of standard methods as e.g. being documented in QIS 4b for the calculation of the solvency capital requirement has been driven forth. Internal models are particularly suitable for covering the risen demands of stakeholders concerning the quality of risk management as the incorporation of sophisticated and flexible mathematical methods can be fulfilled. Another advantage of internal models apart from the improved risk measurement marks the refinement of the risk culture. This might be exemplified by the procedure of rating agencies demanding the existence of an internal model in order for the company to be rated *strong* concerning its risk management. Internal models can be defined as large (high amount of explanatory variables), nonlinear (embedded options), stochastic (modeling future states of nature) systems. In the context of the holistic approach of Basel II and Solvency II an estimation of the balance sheet's forecast distribution is carried out by consulting company models (management rules, provision for premium refunds etc.) as well as stochastic models.

Nevertheless the implementation of internal models implies one thus far not satisfactorily handled issue: the topic of model risk. Without consideration of the latter the capital reserves are determined by the standard approach of risk management. That is portfolio risk is subsumed as the aggregate of the marginal distributions of the risk factors market risk, credit risk and operational risk applying a suitable aggregation method (for a discussion of this topic cf. Rosenberg and Schuermann [2006]) and reporting a risk measure thereof. With the possible utilization of internal models in order to model market risk the risk measure of the latter depends substantially on the concrete specification of the internal model. Thus there does exist a strong relationship between model risk and the resulting market risk which should be accounted for when it comes to the determination of capital reserves. In this context we understand model risk as every risk induced by the choice, specification and estimation of a statistical model.<sup>2</sup>

In order for model risk to be considered as a separate risk factor an operational quantification of the former should be provided. Although some authors like Crouhy et al. [1998] or Cont [2006] made several proposals for an abstract coverage of the topic there does not exist an unambiguous method for the quantification of model risk thus far. In the literature there are basically two approaches dealing with the question of measuring model risk: the bayesian model averaging approach (cf. e.g. Brock et al. [2003]) and the worst-case approach (cf. e.g. Kerkhof et al. [2010]). Although from a practical point of view there is no such thing as obligatory capital charges for induced model risk the Basel Committee (cf. BCBS [1996]) suggests a so-called multiplication factor of three with regard to market risk in order to account for model risk. Stahl [1997] showed that the multiplication factor may be interpreted as the relation of the risk measure of the underlying under different (parametric or non-parametric) distributions. This interpretation corresponds closely to the worst-case approach of measuring model risk. Hence in this paper we follow the idea of Kerkhof et al. [2010] fragmenting model risk into estimation risk, misspecification risk and identification risk and analyze its impact on capital reserves. Our approach features the following new aspects concerning the topic of model risk and capital reserves.

By using real insurance data we do not only analyze the model risk of the underlying but also take the company model into account what, to our knowledge, has not been done before. By describing the structure of an internal model Fig. 1 illustrates this point. Whilst the existing literature does not differentiate between (6) and (9), i.e. the statistical model equals the company model resulting in the assumption that the underlying marks a concrete balance sheet position we take the whole structure of the internal model into account. Concretely we utilize a specific company model, the model for pension liabilities of a large German insurance company, and demonstrate its interaction with the econometric model for the underlying under the aspect of model risk. By taking a broad range of time series models into account which differ in their functional forms we are able to refine the definition of model risk further by discriminating between misspecification risk in functional form and misspecification risk in distribution and are thus enabled to quantify its contribution to overall model risk.

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<sup>2</sup>Note that human failure is captured under operational risk.

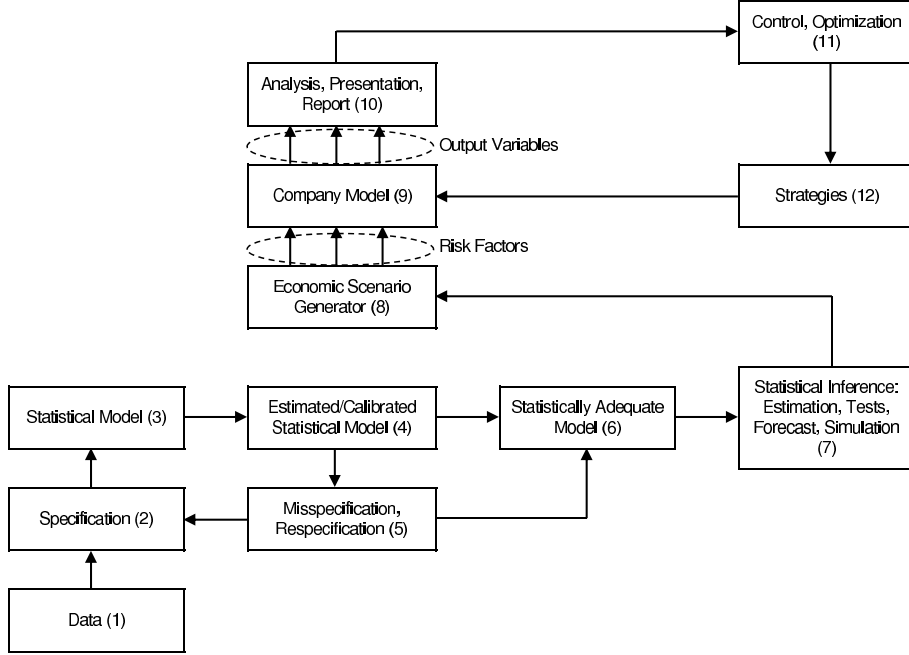


Figure 1: illustrates the structure of an internal model. It is important to emphasize the control circuit on which the process of risk management (consisting of 8-12) is based on. The statistical modeling process (1-7) is preconnected. (5) functions as the feedback of the system.

The paper proceeds as follows. After a formal definition of the various types of model risk the description of the pension model is carried out in section 3. Afterwards we characterize the different econometric models for the underlying in section 4 while developing a classification of univariate time series models and specify, estimate and forecast a broad range of these. The results of the impact of the model choice on the pension liabilities are given in section 5. Following an empirical model specification strategy is carried out and its importance is demonstrated. In section 7 we extend the economic scenario specifying an interest rate model and examine the behavior of model risk in connection with the forecast horizon. Section 8 concludes.

## 2 Measuring Model Risk

Let  $X$  be a random variable forming the stochastic process  $\{X_t\}_{t=-\infty}^{\infty}$  with time index  $t$  being defined on the probability triple  $(\Omega, \mathcal{F}, \mathbb{P})$  consisting of event space,  $\sigma$ -algebra and a probability measure. Assume furthermore that  $X$  is distributed according to some density function  $f_j(X)$ , short  $X \sim f_j(X)$ , where  $\{f_j(X)\}$  with  $j \in \mathcal{J} = (1, \dots, J)$  denotes a set of feasible density functions. With the sample analogue of  $X$  being defined by  $x \equiv \{x_t\}_{t=1}^T$  with  $T < \infty$ ,  $x$  is now assumed to be generated by a so-called data generating process (DGP).

The DGP connects the theoretical distribution of  $X$  with its empirical counterpart by introducing the  $k$ -dimensional parameter space  $\Theta \subseteq \mathbb{R}^k$  which determines the character of the empirical distribution function. The population parameter  $\theta \in \Theta$  marks the point in  $\Theta$  that generated the data leading to the definition of the DGP being given as  $D_x(x|\theta)$ . In a parametrical environment  $\theta$  may then be estimated via consideration of  $f_j(X)$ . If there is no assumption about  $f_j(X)$  nonparametric methods might be applied.

Let further  $\pi_j(p)$  be defined as a risk measure according to the confidence level  $p$ . Famous examples of  $\pi$  are the value-at risk (VaR) being defined as  $VaR_j(p) = \inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) \geq p\}$  and the expected shortfall (ES) being proposed by Artzner et al. [1999]:  $ES_j(p) = E_{\mathbb{P}}(x \in \mathbb{R} | x \geq \inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) \geq p\})$ .  $\pi_j(p)$  might then be defined as the market risk of  $X$  at level  $p$  under the assumption of  $X \sim f_j(X)$ .

Note however that in practical work  $\theta$  and  $f_j(X)$  and thus  $D_x$  are unknown. While the former is typically estimated consulting  $x$ ,  $f_j(X)$  and  $D_x$  are rather specified via assumptions. Consequently the risk measure marks an estimate, too, resulting in an operational definition of market risk.

**Definition 1** (Market Risk). *Let  $X$  be a random variable with  $X \sim f_j(X)$ ,  $D_x(x|\theta)$  and sample analogue  $x$ . Then the market risk of  $X$  concerning the confidence level  $p$  is given by  $\hat{\pi}_j^{(1)}(p) = \inf\{x \in \mathbb{R} | \int_{-\infty}^x f_j(x|\hat{\theta})dx \geq p\}$  for the VaR and by  $\hat{\pi}_j^{(2)}(p) = E_{\mathbb{P}}(x \in \mathbb{R} | x \geq \inf\{x \in \mathbb{R} | \int_{-\infty}^x f_j(x|\hat{\theta})dx \geq p\})$  for the ES.<sup>3</sup>*

Taking the consideration of estimation uncertainty into account leads to the definition of estimation risk as a component of model risk. Since the risk measure marks a point estimate of a quantile of its probability distribution a confidence interval  $[\hat{\pi}_j \pm \eta_j(\alpha)]$  for the estimate at level  $1 - \alpha$  can be defined. Via the functional delta method eta can be derived as  $\eta_j(\alpha) = z_{1-\alpha/2} \hat{\sigma}_j$ , where  $\hat{\sigma}_j = \sqrt{p(1-p)/T \cdot f_j(\hat{\pi}_j(p))^2}$ . The quotient of the upper bound of the confidence interval and the point estimate yields a definition for estimation risk.

**Definition 2** (Estimation Risk). *Let  $\Pi_j(p, \alpha) = \hat{\pi}_j(p) + \eta_j(\alpha)$  be the upper bound of the  $1 - \alpha$  confidence interval of the risk measure's point estimate. Estimation risk is then given by  $ER_j = \Pi_j(p, \alpha) \cdot \hat{\pi}_j(p)^{-1}$ .*

Consider now the case where the assumed distribution of  $X$  does not correspond to its true distribution, i.e.  $D_x(x|\theta)$  is misspecified. Indexing the assumed distribution by  $j = 1$  and the true distribution by  $j = 2$  yields a formulation of misspecification risk. If  $f_2(X)$  is taken to be some estimate of the empirical density function a measure of total model risk corresponding to the Basel multiplication factor can be formulated.

**Definition 3** (Total Model Risk). *Let  $\Pi_2(p, \alpha)$  be the upper bound of the  $1 - \alpha$  confidence interval of the risk measure's point estimate under the empirical density function while  $\hat{\pi}_1(p)$  denotes the market risk measure under an assumed density function  $f_1$ . Total model risk in correspondence to  $f_1$  can then be derived as  $MR = \Pi_2(p, \alpha) \cdot \hat{\pi}_1(p)^{-1}$ .*

We may now look at the DGP in more detail. In order for the DGP to be made operable in empirical work  $D_x(x|\theta)$  is mapped to the generic econometric equation  $x_t = h(z_t|\theta) + \varepsilon_t$  where  $z_t = (x_{t-1}, \dots, x_0, y_t, \dots, y_0) = (\tilde{x}_{t-1}, \tilde{y}_t)$  contains all kinds of endogenous ( $\tilde{x}_{t-1}$ ) and/or exogenous ( $\tilde{y}_t$ ) explanatory variables,  $h(\cdot)$  describes the functional form of the relationship and  $\varepsilon_t$  denotes an error term. If now  $h(z_t|\theta) = 0$ ,  $\hat{\pi}_1(p)$  corresponds to the assumed distribution for  $\varepsilon$  while  $\hat{\pi}_2(p)$  is interpreted as the risk measure corresponding to the empirical density function of  $\{x\} = \{\varepsilon\}$  resulting automatically in the definition of model risk given by Def.3. If however  $h(z_t|\theta) \neq 0$  we may differentiate the definition of model risk even further into misspecification risk in distribution and misspecification risk in functional form. For the derivation of the former consider the case where  $x_t = h(z_t|\theta) + \varepsilon_t$  with  $h(z_t|\theta) \neq 0$  and  $\{\varepsilon\} \sim f_2(\varepsilon)$ . Further denoting the resulting density function of  $x$  by  $f_3(X)$  leads to the following definition.

**Definition 4** (Model Risk in Distribution). *Let  $\Pi_3(p, \alpha)$  be the upper bound of the  $1 - \alpha$  confidence interval of the risk measure's point estimate with the DGP of the underlying being given as  $x_t = h(z_t|\theta) + \varepsilon_t$  with*

<sup>3</sup>If not mentioned otherwise  $\pi_j$  describes the VaR in the upcoming analysis. In case we reported solely the VaR risk measure the results for the ES did not differ profoundly.

$h(z_t|\theta) \neq 0$  and  $\{\varepsilon\} \sim f_2(\varepsilon)$  while  $\hat{\pi}_1(p)$  denotes the market risk measure under an assumed density function  $f_1$ . Model risk in distribution in correspondence to  $f_1$  can then be derived as  $MR_{dis} = \Pi_3(p, \alpha) \cdot \hat{\pi}_1(p)^{-1}$ .

Straightforwardly we can now define model risk with regard to the functional form.

**Definition 5** (Model Risk in Functional Form). *Let  $\Pi_2(p, \alpha)$  be the upper bound of the  $1 - \alpha$  confidence interval of the risk measure's point estimate under the empirical density function while  $\hat{\pi}_3(p)$  denotes the risk measure's point estimate with the DGP of the underlying being given as  $x_t = h(z_t|\theta) + \varepsilon_t$  with  $h(z_t|\theta) \neq 0$  and  $\{\varepsilon\} \sim f_2(\varepsilon)$ . Model risk in functional form in correspondence to  $h(z_t|\theta)$  can then be derived as  $MR_{ff} = \Pi_2(p, \alpha) \cdot \hat{\pi}_3(p)^{-1}$ .*

Note that the the components of model risk are connected via the relationship  $MR = MR_{dis} \cdot MR_{ff} / ER_2$ . Since so far solely the underlying has been regarded in the following we should look at model risk concerning the company model. With the pension liabilities being chosen as an example of a company model the underlying  $X$  and the liabilities  $L$  are connected via the relationship  $L = g(X)$  where  $g(\cdot)$  denotes a continuously differentiable function.<sup>4</sup> Then the density of  $L$  with respect to  $f_j(X)$  is given by  $\tilde{f}_j(L) = f_j(X) \cdot |dh(L)/dL|$  where  $h(L) = g^{-1}(L)$ . Hence the various definitions of market risk and model risk (total, in distribution and in functional form) can easily be transferred to the company model by substituting  $f_j(X)$  for  $\tilde{f}_j(L)$ . Note however that we cannot define an estimation error in this setting since the company model cannot be handled like an econometric model.

### 3 The Pension Model

The model's objective is to calculate path-dependent future pension liabilities  $L$ . In general these pension liabilities depend on many different factors including economic variables. As the development of the economy goes along with great uncertainties, future pension liabilities exhibit a stochastic behavior. According to IFRS the two most important explanatory variables for the pension liabilities from an economic point of view are the inflation rate  $\mathbb{I}$  and the interest rate  $\mathbb{Y}$ . The reason for the former to play a great role in the determination of  $L$  is the fact that it is used as a discount rate whereas the latter functions as an adjustment for the obligations in terms of salary rates.

In practice the calculation of pension liabilities is a very complex procedure as  $L$  further depends on a wide range of other factors, i.e. mortality risks, contractual details, possible cancelations etc., which additionally are very likely to differ between a number of  $v = 1, \dots, P$  pensioners. With those factors being pooled in  $\Theta$ , the pension liability  $L_k$  of a portfolio  $k \in K = \{\text{Actives, Vested Benefits, Pensioners}\}$  at time  $t$  can be formalized as

$$L_{jkt} = \sum_{v=1}^P L_{vjkt} = g(\mathbb{I}_{jkt}, \mathbb{Y}_{jkt}; \Theta_v), \quad (3.1)$$

where  $j = (1, \dots, N)$  denotes the number of paths. 3.1 is called the *Pension Valuation System (PVS)*. Note that for  $t > t^*$  where  $t^*$  denotes the current point in time,  $L_{jkt}$  is a random variable since for every random variable  $X$  and measurable function  $g(\cdot)$ ,  $g(X)$  is a random variable as well. Hence in order to analyze the stochastic behavior of the pension liabilities appropriately a sufficiently large amount of paths of  $L_j = g(\mathbb{I}_j, \mathbb{Y}_j)$  have to be simulated. Whilst the simulation can easily be carried out for  $\mathbb{I}$  and  $\mathbb{Y}$ , the calculations of the corresponding  $L$  are too computationally intensive to be carried out for every single path. In other words the function  $g(\cdot)$  is too complex to be utilized for the necessary amount of operations.

<sup>4</sup>A concrete specification of  $g(\cdot)$  is given in section 3.

Thus the objective is to specify a function  $f(\cdot)$  that approximates  $g(\cdot)$  sufficiently well. Once this function is found and its parameters are calibrated, the liabilities can be fitted via application of

$$\hat{L}_{jkt} = \hat{f}(\mathbb{I}_{jkt}, \mathbb{Y}_{jkt}). \quad (3.2)$$

$f(\cdot)$  will be called the *Pension Function (PF)* with  $\hat{f}(\cdot)$  denoting its calibrated counterpart. The calibration of the *PF* is applied by using subsets  $I \subseteq \mathbb{I}$ ,  $Y \subseteq \mathbb{Y}$  of the simulated paths. Those subsets are called sample points and they are chosen as to represent  $\mathbb{I}$  and  $\mathbb{Y}$  properly. Once the sample points are determined the corresponding pension liabilities can be calculated by the *PVS* via application of  $L_{ikt} = g(I_{ikt}, Y_{ikt}; \Theta_v)$ , where  $i = (1, \dots, M)$  denotes the number of sample points. Note that  $M$  should be chosen such that the reduction of the number of paths from  $N$  to  $M$  leads to an operable calculation of  $L$ . After this has been done  $f(\cdot)$  can be calibrated using the input data  $(L_{ikt}, I_{ikt}, Y_{ikt})$ .

The above procedure is called dynamic transformation approach. This approach can be described as a process in which an existing external system is run using a small number of prescribed deterministic paths and the output generated is then transformed and used to simulate a much larger set of path-dependent results. Hence the dynamic transformation approach leverages an existing single-path without requiring it to be run for all paths. The objective of the dynamic transformation model is to produce results that are consistent with what the external system would produce if it were run over all paths. The leveraging is necessary as the calculation of the pension liabilities entails a very complex operation. It should be made clear that this paper solely deals with the influence of the econometric model on the pension liabilities concerning model risk, i.e. the difficulties occurring in the selection of model points and calibration of the pension function are not considered in this context and are open to future examination.

Concretely the *PF* is given as

$$L_{i,k} = f_k(I_{i,k}, Y_{i,k}) = \beta_{0,k} \cdot (1 + \beta_{1,k} \cdot I_{i,k})^{\beta_{2,k}} \cdot (1 + \beta_{3,k} \cdot Y_{i,k})^{-\beta_{2,k}}, \quad (3.3)$$

where  $(L_k, I_k, Y_k)$  are vectors of dimension  $(M \times 1)$  respectively,  $\beta_k = (\beta_{0,k}, \dots, \beta_{3,k})$  is a  $(4 \times 1)$  vector of coefficients,  $k \in \{1, \dots, K\}$  and  $i = (1, \dots, M)$ . Fig.3 gives an idea of the shape of the function.<sup>5</sup> The function

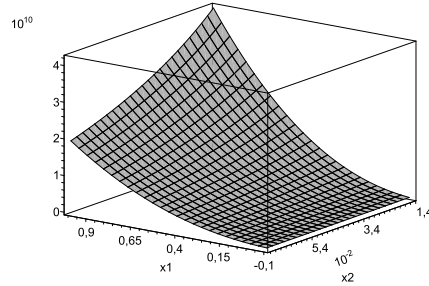


Figure 2: illustrates the shape of the pension function.  $x_1$  describes the inflation rate, the interest rate is given by  $x_2$  and  $L$  is displayed on the  $z$ -axis. Note that the combination of high inflation rates and low interest rates leads the function to rise quickly.

can be interpreted as follows.  $\beta_{0,k}$  represents the pension value if both  $I_{i,k} = Y_{i,k} = 0$ , that is if there is no inflation and the interest rate takes on zero.  $\beta_{1,k}$  and  $\beta_{3,k}$  are the respective adjustment factors for inflation and

<sup>5</sup>Due to reasons of concealment the  $\beta_k$  could not be reported in this paper.

interest rates whilst  $\beta_{2,k}$  denotes a discount factor. Furthermore it should be mentioned that the time offsets chosen as to represent the appropriate duration of the inflation rate and the interest rate amount to  $t_1 = 10$  and  $t_2 = 17$  years respectively. Thus  $\mathbb{I}_{j_1}$  ( $Y_{j_2}$ ) reflects the expected 10-year (17-year) ahead inflation rate (interest rate) for the  $j$ th path.

With the error between liabilities and the function values being defined as  $\varepsilon_k$  3.3 can be calibrated by application of

$$\min_{\beta_k} \|\varepsilon_k\|_p \quad \text{with} \quad \varepsilon_{i,k} = L_{i,k} - \beta_{0,k} \cdot (1 + \beta_{1,k} \cdot I_{i,k})^{\beta_{2,k}} \cdot (1 + \beta_{3,k} \cdot Y_{i,k})^{-\beta_{2,k}},$$

where  $p$  should be chosen appropriately. The minimization algorithm works as a combination of grid search and hill-climb method.

## 4 The Inflation Models

In order not to further complicate the procedure the economic scenario generator solely deals with modeling the inflation rate for a start. As far as the interest rate is concerned scenarios having been developed internally by the insurance company are utilized (for a brief overview of the interest rate scenario's distribution cf. Fig.4).<sup>6</sup> This is legitimized by considering the position of the inflation rate at the top of the cascade in the benchmark

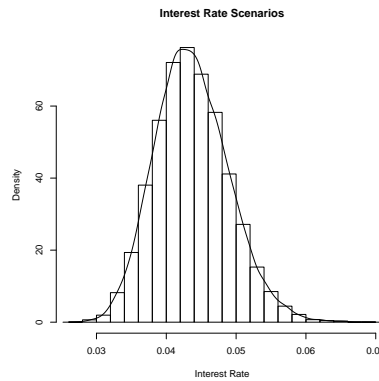


Figure 3: describes the histogram and density estimation of the interest rate scenarios generated by the insurers internal economic scenario generator.

Wilkie model (Wilkie [1995]) mirroring its particular importance. In case a misspecified inflation model is utilized the misspecification error transmits throughout the whole system. This means that dealing with the inflation model should be of the highest priority when it comes to specifying an economic scenario generator that is to outperform the Wilkie model.

In order to carry out a consistent procedure of model specification a hierarchy of the univariate time series models being used in practice is very helpful. In the first level we discriminate between linear and nonlinear models. Recall that by the Wold decomposition any zero-mean purely non deterministic stationary process  $\{y_t\}_{t=1}^T$  can be written in the form

$$y_t = \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \quad (4.1)$$

<sup>6</sup>In chapter 7 the relation between the inflation model and interest rates is further analyzed.



where  $\sum_i \|\Psi_i\|^2 < \infty$  and  $\{\varepsilon_t\}_{t=1}^T$  is a stationary sequence of centered and uncorrelated random variables with common variance  $\Sigma$ . A process  $\{y_t\}_{t=1}^T$  is said to be linear when  $\{\varepsilon_t\} \stackrel{iid}{\sim} (0, \Sigma)$  in 4.1. Otherwise the process can be declared nonlinear. Note that the nonlinearity can occur in the mean as well as in the volatility.

This thought leads to the discrimination of three classes of time series models in our model selection procedure: linear models, nonlinear models in the mean and nonlinear models in the volatility. Fig.4 illustrates the classification of time series models. Whereas the major part of the existing linear models can be subsumed

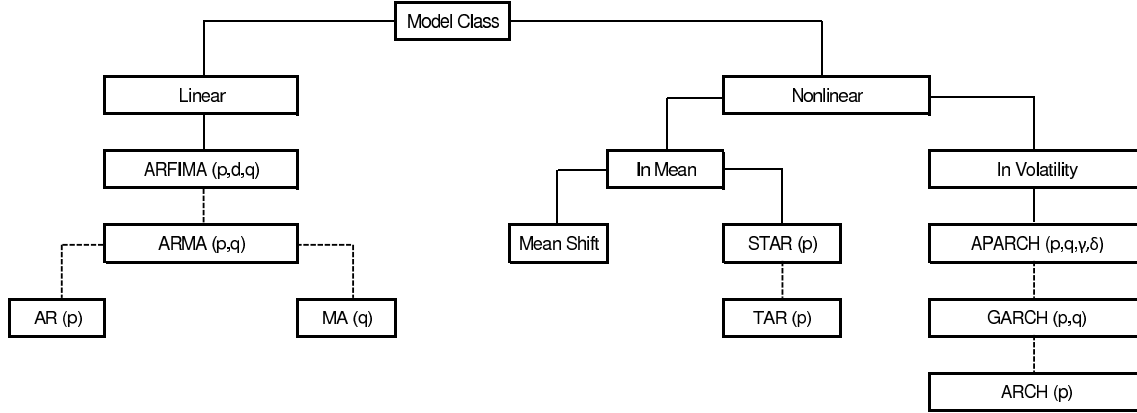


Figure 4: describes the classification of univariate time series models. The dotted lines signify nested relationships.

under the class of autoregressive fractionally integrated moving average models, short  $ARFIMA(p, d, q)$ , the class of nonlinear models is not as homogenous.

**ARFIMA models.** The  $ARFIMA(p, d, q)$  model can be defined as

$$\Theta(L)^{-1} \Phi(L) (1-L)^d y_t = \varepsilon_t, \quad (4.2)$$

where  $\{y_t\}_{t=1}^T$  describes the time series of interest and  $\{\varepsilon_t\}_{t=1}^T$  forms a white noise process.  $\Theta(L) = 1 - \theta_1 L - \theta_2 L^2 - \dots - \theta_q L^q$  and  $\Phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$  are polynomials of degrees  $q$  and  $p$  with  $L^i$  being defined as the backshift operator such that  $L^i \cdot y_t = y_{t-i}$ .  $d$  describes the fractional differencing parameter. Note that by setting zero restrictions on the respective parameters 4.2 nests a broad range of linear time series models like autoregressive and moving average processes.

For  $d \in \mathbb{Z}_0^+$  the resulting model belongs to the well-known class of  $ARIMA$  models having elaborately been examined by Box and Jenkins [1976]. The estimation procedure in this model class is usually be carried out by testing for unit roots first. Some formal tests for distinguishing between  $d = 0$  and  $d = 1$  exist (see e.g. Dickey and Fuller [1979] or Kwiatkowski et al. [1992]). After having differenced the process  $d$  times the resulting  $ARMA$  model can be estimated by several methods such as conditional sum of squares or maximum likelihood estimation (MLE). Setting the order of the moving-average (autoregressive) part equal to zero leads to the class of autoregressive (moving-average) processes.

For many empirical time series however taking the first or the second difference seems somewhat exaggerated whereas not differencing the series at all does not yield stationarity. Therefore fractional differencing was proposed by Hosking [1981] and Granger and Joyeux [1980] with  $-\frac{1}{2} < d < \frac{1}{2}$  being able to model long-range dependencies between (economic) variables adequately.  $d$  can be estimated via maximum likelihood methods (cf. e.g. Yajima [1985]). Beran [1995] additionally suggests an estimation method in case of nonstationary long-memory ( $d \geq 0.5$ ). Hassler and Wolters [1995] examined the inflation rates of five countries finding ev-



idence of long memory in each series. For a more detailed description of the estimation procedure cf. also section 6.

**Mean shift models.** The simplest form of nonlinearity can be described by mean shift models. By rewriting a time series  $y_t = \mu_t + \varepsilon_t$  as the sum of a deterministic part  $\mu_t$  and a stochastic part  $\varepsilon_t$  the former can be modelled as  $\mu_t = \mu_1 + \sum_{i=1}^n \lambda_i \cdot I(l_i < t \leq l_{i+1})$  where  $n$  denotes the number of breaks,  $l_i$  are the break points,  $I(\cdot)$  denotes the indicator function and  $\lambda_i = \mu_{i+1} - \mu_i$ . Note that structural changes in the mean are a typical example of the occurrence of spurious long memory (cf. Diebold and Inoue [2001] or Engle and Smith [1999]). By neglecting the mean shifts the estimation of the fractional differencing parameter  $d$  might be biased quite heavily. That is why Hsu [2005] proposed to first determine the number of break points in the model and thereafter estimate the *ARFIMA* parameters and the time of the breaks simultaneously. Whereas the former is done via application of the *LIC* information criterion described in Lavielle and Moulines [2000] the estimation is carried out by a modified local Whittle method.

**STAR models.** One of the most prominent regime-switching model marks the smooth transition autoregressive (*STAR*) model introduced by Chan and Tong [1986] and popularized by Granger and Teräsvirta [1993] and Teräsvirta [1994]. It is given by

$$y_t = (\phi_{0,1} + \phi_{1,1}y_{t-1} + \dots + \phi_{p_1,1}y_{t-p_1})(1 - G(y_{t-1}; \gamma, c)) + (\phi_{0,2} + \phi_{1,2}y_{t-1} + \dots + \phi_{p_2,2}y_{t-p_2})G(y_{t-1}; \gamma, c) + \varepsilon_t. \quad (4.3)$$

Thus the *STAR* model is given by two autoregressive regimes connected by the transition function  $G(\cdot) \in [0, 1]$  plus a white noise error term. If  $G(\cdot)$  is a continuous function the transition between the two regimes is carried out smoothly. The transition occurs once the threshold value  $c$  is passed such that  $G(c; \gamma, c) = 0.5$ . Popular choices for the transition function are the exponential function (*ESTAR*) or the logistic function (*LSTAR*). In the latter case  $G(y_{t-1}; \gamma, c) = (1 + \exp(-\gamma(y_{t-1} - c)))^{-1}$ . Note that  $\gamma$  determines the smoothness of the transition. If e.g.  $\gamma$  is very large the change of  $G(\cdot)$  from 0 to 1 becomes almost instantaneous at  $y_{t-1} = c$  whereas for  $\gamma \rightarrow 0$ ,  $G(\cdot)$  converges to a constant and 4.3 reduces to a linear model. In the former case one speaks of a threshold autoregressive (*TAR*) model. For an example of an *ESTAR* application on the inflation rate cf. Gregoriou and Kontonikas [2006] while an example for an *LSTAR* can be found in Huh [2002].

4.3 can be estimated via nonlinear least squares (*NLS*). Special care has to be taken concerning the choice of the starting values for the optimization algorithm as well as the estimation of the smoothness parameter  $\gamma$  in the transition function due to reasons of identification.

**APARCH models.** The third class of time series models to be examined is the group of regime-switching models for the volatility called general autoregressive conditional heteroscedasticity (*GARCH*) models. The latter are very popular among practitioners (cf. e.g. Karanasos et al. [2004]) as they are not only capable of describing stylized facts of financial time series such as excess kurtosis or fat-tailedness but also incorporate the concept of volatility clustering meaning that periods of large movements in prices alternate with periods during which prices hardly change. The general asymmetric power *GARCH* (*APARCH*) was introduced by Ding et al. [1993] and is defined as

$$y_t = \mu + \sum_{i=1}^p a_i y_{t-i} + \sum_{j=1}^q \varepsilon_{t-j} + \varepsilon_t, \quad (4.4)$$

$$\begin{aligned} \varepsilon_t &= h_t^{1/\delta} \cdot v_t, \\ h_t &= \omega + \sum_{k=1}^K \alpha_k (|\varepsilon_{t-k}| - \psi_k \cdot \varepsilon_{t-k})^\delta + \sum_{l=1}^L \beta_l \cdot h_{t-l}, \end{aligned} \quad (4.5)$$

where  $\mu$  and  $\omega$  are constants,  $a$ ,  $\alpha$  and  $\beta$  are vectors of coefficients and  $\{v_t\} \stackrel{iid}{\sim} (0, \Sigma)$ . Obviously  $\varepsilon_t$  is now no longer assumed to be homoscedastic but conditionally heteroscedastic as  $E[\varepsilon_t^2 | \Omega_{t-1}] = h_t$  for  $\delta = 2$  with  $\Omega_{t-1}$  describing the information set of all relevant information up to and including time  $t - 1$ .

$\psi$  reflects the so-called leverage effect taking into account that positive and negative shocks might have a different impact on the conditional volatility of the process. By rewriting 4.5 for  $\delta = 2$  as

$$h_t = \omega + \sum_{k=1}^K [\alpha_k(1 - \psi_k)^2 + 4\alpha_k\psi_k \cdot I(\varepsilon_{t-k} < 0)] \cdot \varepsilon_{t-k}^2 + \sum_{l=1}^L \beta_l \cdot h_{t-l}$$

with  $I(\cdot)$  denoting the indicator function it can be seen that negative shocks have an impact of  $\alpha_k(1 - \psi_k)^2 + 4\alpha_k\psi_k$  on the conditional variance, while for positive shocks the impact reduces to  $\alpha_k(1 - \psi_k)^2$ . Finally  $\delta > 0$  mirrors the parameter of the Box-Cox transformation. Note that by setting (zero) restrictions on  $a$ ,  $\alpha$ ,  $\beta$ ,  $\psi$  and/or  $\delta$  several nested models can be specified (cf. Bollerslev [2008]) while for all the reported models in section 5,  $\delta$  was set to 2 specifying a *GJR-GARCH* (cf. Glosten et al. [1993]) for  $\psi \neq 0$ . 4.4 can be estimated by MLE. The specified models and its parameters are given in Tab.1.

Equation	Model	Parameter										Notation
		p	q	d	$p_1$	$p_2$	c	$\gamma$	K	L	$\psi$	
4.2	M1	1	0	0	-	-	-	-	-	-	-	Wilkie, AR(1)
	M2	2*	2*	0	-	-	-	-	-	-	-	ARMA(2,2)
	M3	0	0	0.178**	-	-	-	-	-	-	-	ARFIMA(0,d,0)
	M4	2*	2*	0.118**	-	-	-	-	-	-	-	ARFIMA(p,d,q)
	M5	0	0	0.083**	-	-	-	-	-	-	-	HSU
4.3	M6	-	-	-	1	1	0.005**	40**	-	-	-	STAR(1,c, $\gamma$ )
	M7	-	-	-	13*	13*	-0.012**	40**	-	-	-	STAR(p,c, $\gamma$ )
4.4	M8	0	0	0	-	-	-	-	1	0	0	ARCH(1)
	M9	0	0	0	-	-	-	-	4*	0	0	ARCH(K)
	M10	0	0	0	-	-	-	-	1	1	0	GARCH(1,1)
	M11	1	0	0	-	-	-	-	1	0	0	ARMA(1,0)-ARCH(1)
	M12	1	0	0	-	-	-	-	2*	0	0	ARMA(1,0)-ARCH(K)
	M13	1	1	0	-	-	-	-	1	1	0	ARMA(1,1)-GARCH(1,1)
	M14	0	0	0	-	-	-	-	1	0	0.083**	APARCH(1)
	M15	0	0	0	-	-	-	-	1	1	0.102**	APARCH(1,1)

Table 1: offers an overview of the specified models. \* signifies that the respective lag order has been chosen via information criteria. \*\* marks estimated values. Note that in M6 and M7  $\gamma$  was respectively estimated to equal 40 signifying that the regime-switch is not carried out smoothly. In fact a threshold autoregressive (TAR) model is specified. The model specifications reported here are the most striking ones regarding its impact on the pension function. We examined a broad range of further specifications which can be reported upon request.

## 5 Simulation Results

### 5.1 Inflation Models

The modeling of the inflation rate has been carried out by using monthly US inflation data for the period 01/1954 until 02/2010 taken from Datastream. The time series and its empirical density estimate is plotted in Fig.5.

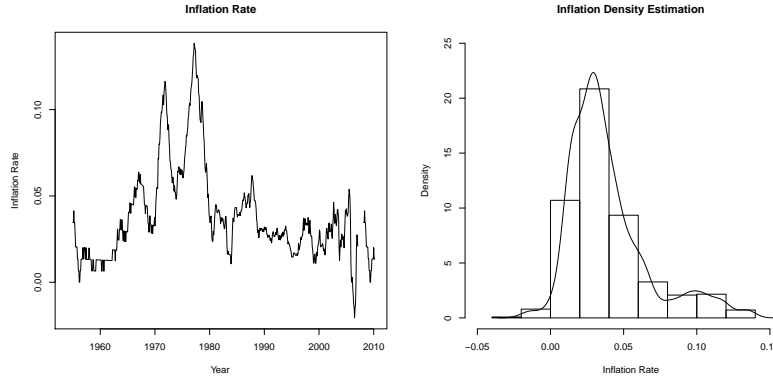


Figure 5: plots the monthly US inflation rate being calculated as the difference of the log consumer price indexes in regard to the respective value from the previous year and the corresponding empirical density estimate.

The *ADF* test (cf. Dickey and Fuller [1979]) as well as the *KPSS* test (cf. Kwiatkowski et al. [1992]) indicate the series to be  $I(1)$  which is why henceforward its first difference is utilized. The procedure concerning the simulation of the inflation rates is given as follows. For each of the  $K = 15$  models  $M_k$ ,  $k = 1, \dots, K$ , the parameters are estimated. Following forecast values  $\hat{y}_{t+h}$  with  $h = 1, \dots, H$  are derived, where the forecast period is chosen to equal  $H = 118$ . This value accounts for the fact that 3.3 necessitates the 10-year ahead inflation rate while having monthly data up to 02/2010. The forecast values are then given by

$$\hat{y}_{t+h} = E(y_{t+h} | \Omega_{t+h-1}) + \varepsilon_{t+h}, \quad h = 1, \dots, H \quad (5.1)$$

where  $\Omega_{t+h-1}$  is the information set consisting of all relevant information up to and including time  $t + h - 1$ . The yearly inflation rate is then given by the year's mean value. This procedure is replicated  $N = 10,000$  times for each of the  $K = 15$  models yielding the empirical distributions which are summarized in Tab.4 (cf. section B).

The first striking result marks the fact that the differences of the inflation's distributions mainly focuses on its tails. Whereas the central part of the distributions is surprisingly homogenous the more extreme quantiles and the range differ considerably. This is especially driven by those models belonging to the class of *GARCH* processes (i.e. M10, M13 and M15). Although these models forecast rather plausible 10-year ahead inflation rates of approximately 2% in the mean its worst case scenarios of e.g. 170% deflation do not seem to be very realistic.

An explanation for these features can be given by more thoroughly looking at the autocorrelation function of the *GARCH*(1,1) process. Bollerslev [1986] and Bollerslev [1988] showed that the  $k$ th autocorrelation of

the squared errors in the  $GARCH(1, 1)$  process is given by

$$\rho_1 = \alpha_1 + \frac{\alpha_1^2 \beta_1}{1 - 2\alpha_1 \beta_1 - \beta_1^2} \quad (5.2)$$

$$\rho_k = (\alpha_1 + \beta_1)^{k-1} \rho_1 \quad \text{for } k = 2, 3, \dots \quad (5.3)$$

Note that the decay factor of 5.3 is  $\alpha_1 + \beta_1$ . If the sum is close to 1 the autocorrelations will decline only very gradually (although an exponential decline is still given). In our case the sum of the estimated coefficients from the respective  $GARCH$  models are in each of the three cases very close to 1 i.e. the  $GARCH$  models feature slowly decaying autocorrelation functions. This leads to the result that draws of exceptionally high error terms during the simulation process 5.1 hardly decline in this model class explaining the extreme scenarios. Note that the fact of the sum of the estimated parameters in  $GARCH(1, 1)$  models being close to 1 is commonly found in empirical research. E.g. Taylor [1986] estimated  $GARCH(1, 1)$  models for 40 different financial time series finding in all but six cases that  $0.97 \leq \alpha_1 + \beta_1 < 1$ .

It should furthermore be mentioned that Bollerslev [1986] and Bollerslev [1988] conditioned 5.2-5.3 on the validity of  $(\alpha_1 + \beta_1)^2 + 2\alpha_1^2 < 1$  signifying that the kurtosis of  $\varepsilon_t$  is finite. If however this cannot be maintained, which is the case in our analysis, Ding and Granger [1996] showed that for  $\alpha_1 + \beta_1 < 1$  and  $(\alpha_1 + \beta_1)^2 + 2\alpha_1^2 \geq 1$  the  $GARCH(1, 1)$  model is still covariance stationary with infinite fourth moment. In this case 5.3 is approximately valid with  $\rho_1 \approx \alpha_1 + \beta_1/3$ . Note also that the  $HSU$  model (M5) features a lower mean than the other models. This is due to the fact that we determined  $n = 1$  break point via the  $LIC$  criterion at  $t^* = 302$  which corresponds to 02/1979. By looking at Fig.5 it becomes clear that after  $t^*$  the trend in the inflation rate is declining what explains the lower mean of M5 even over 10,000 replications.

The risk measures being defined in section 2 are summarized in Tab.2. By first concentrating on the esti-

Model	$\hat{\pi}_1$	$ER_1$	$ER_2$	$MR_{dis}$	$MR_{ff}$	$MR$
M1	0.15	1.088	1.022	0.988	0.845	0.816
M2	0.15	1.085	1.024	0.997	0.807	0.786
M3	0.24	1.096	1.028	1.016	0.499	0.493
M4	0.09	1.068	1.022	1.057	1.362	1.408
M5	0.15	1.286	1.025	1.018	0.814	0.808
M6	0.14	1.098	1.024	1.000	0.874	0.854
M7	0.14	1.079	1.023	1.015	0.819	0.813
M8	0.12	1.093	1.029	0.994	1.029	0.994
M9	0.14	1.116	1.029	0.868	1.029	0.868
M10	0.21	1.189	1.029	0.589	1.029	0.589
M11	0.14	1.085	1.024	0.974	0.893	0.849
M12	0.15	1.088	1.025	0.975	0.877	0.833
M13	0.26	1.278	1.029	3.231	0.134	0.421
M14	0.12	1.077	1.029	1.055	1.029	1.055
M15	0.36	1.568	1.029	0.375	1.029	0.375

Table 2: returns measures of market risk, estimation risk, model risk in distribution, model risk in functional form and total model risk being defined in section 2 for each of the models with  $p = 0.99$  and  $\alpha = 0.05$ .

mation risk with regard to  $f_1(X)$  it becomes clear that for the majority of the models the estimation error lies somewhere between 5 and 10 percent. The  $GARCH$  model class again forms an exception with estimation risks

up to almost 57% which naturally can be attributed to its near-integratedness and high volatilities. Together with *M3* those are also the models featuring low multiplication factors being displayed in the last column. Note that in many popular examples  $MR > 1$  as the underlying marks a financial market variable exhibiting the stylized fact of fat-tailedness. This results in the empirical density function having a higher kurtosis than most of the standard parametric models which is why  $\pi_2(p) > \pi_1(p)$  with the resulting multiplication factor exceeding 1. Inflation however is not a monetary but a real variable usually not featuring these stylized facts. Hence in many cases (except for *M4* and *M14*) the parametric distribution possesses a much higher kurtosis than its empirical counterpart. Thus our results mirror the following trade-off. Those models implying a low market risk are penalized by a multiplication factor greater than 1 concerning the capital reserves. If a model reports a high market risk it is compensated by a multiplication factor smaller 1.

Another crucial aspect is the fact that model risk in functional form marks the main factor for the determination of total model risk. By looking at the last three columns of Tab.2 it becomes clear that except for the *GARCH* model class  $MR_{dis}$  is very close to 1 indicating low explanatory power for  $MR = MR_{dis} \cdot MR_{ff} / ER_2$  which is a rather intuitive result. Note further that there are five models exhibiting the same model risk in functional form of 2.9%. For those models  $h(z_t|\theta) = 0$  in  $x_t = h(z_t|\theta) + \varepsilon_t$  which means that there are no short-term dependencies in the *DGP*. Hence model risk in functional forms reduces to the estimation error concerning  $\hat{\pi}_3(p)$ , i.e.  $\Pi_2(p, \alpha) \cdot \hat{\pi}_3(p) = \eta_2(\alpha)$ . This also becomes clear by looking at column 3 where the estimation risk concerning  $x$  with  $\varepsilon \sim f_2(\varepsilon)$  is described.<sup>7</sup> In other words  $f_3(X) = f_2(\varepsilon)$  results in model risk in distribution exclusively determining total model risk.

Hence we can conclude that market risk as well as model risk differs substantially between the various econometric models. Whereas theoretically there is a trade-off between market risk and model risk as both depend on the functional form and the distribution of the underlying practically the Basel multiplication factor is fixed which leads to the fact that the model inducing the lowest market risk implies the lowest capital reserves. The implication of this result in terms of monetary values is outlined in the next section.

## 5.2 Pension Liabilities

Once the inflation scenarios have been determined the corresponding pension liabilities are calculated by 3.3 via the calibrated parameters  $\hat{\beta}$ . The resulting distributions of  $L_k$  are summarized in Tab.5 (cf. section B). Note that due to reasons of concealment the respective difference to the benchmark Wilkie model (*M1*) instead of the absolute values are reported.

Again the models of the *GARCH* class feature distinct differences in the tails in comparison with the other models. By recapitulating the shape of the pension function (cf. Fig.3) this result should not be surprising. Remember that the *GARCH* inflation scenarios exhibit values in its right tail that are well above 0.4 which is exactly the area where 3.3 increases rapidly. As the pension function is leveraged by the inflation scenarios the center of the distributions differ slightly more than the scenarios itself. Nevertheless the most striking deviations are once more found in the distributions' tails.

By looking at Tab.5 it can be seen that the values for the VaR differ enormously. Note that the values are reported in million € implying a discrepancy of the model with the lowest VaR (*M4*) and the model with the highest VaR (*M15*) of 5,717.4 million €. Of course, one might argue that common sense allows the exclusion of the *GARCH* model class but then still the difference adds up to 2,074.3 million € (*M4* vs. *M3*). Regarding the expected shortfall the differences in the pension liabilities are even more striking going from 18,359.6 million € without exclusion of the *GARCH* to 2,497.2 million € without consideration of *M10*, *M13* and *M15*. Generally it becomes clear that both a high range and excess kurtosis in the econometric models produces the kurtosis in the pension liabilities' distribution to rise resulting in large values for the VaR and ES.

As was argued in section 2 the market risk of  $L$  is given by  $\tilde{\pi}_j(p) = \inf\{L \in \mathbb{R} | \int_{-\infty}^L \tilde{f}_j(L|\hat{\theta}; \hat{\beta}) dL \geq p\}$  with  $\tilde{f}_j(L) = f_j(X) \cdot |dh(L)/dL|$ . 3.3 automatically yields  $h(L) = \beta_1^{-1}((L/\beta_0(1 + \beta_3 Y)^{-\beta_3})^{1/\beta_1} - 1)$  which is why

<sup>7</sup>Note that the fact that  $ER_2 = 1.029$  for *M13* is an artefact caused by rounding. It is not perfectly equal to  $\eta_2(\alpha)$  concerning  $h(\cdot) = 0$ .

$|dh(L)/dL| = (\beta_1\beta_2L)^{-1}(L/\beta_0(1 + \beta_3Y)^{-\beta_3})^{1/\beta_1}$ . Denoting the absolute difference of  $\tilde{\pi}_i(p)$  and  $\tilde{\pi}_j(p)$  where  $i, j \in \mathcal{J}$  by  $\tilde{\pi}_{ij}$  we may now define the different types of model risk in terms of monetary values. Hence total model risk is given by  $mr = \tilde{\pi}_{12}$ , model risk in distribution is defined as  $mr_{dis} = \tilde{\pi}_{13}$  and  $mr_{ff} = \tilde{\pi}_{23}$  describes model risk in functional form.

Tab. 3 returns the resulting values in million € with market risk being again measured as the difference to the Wilkie model  $M1$ . It is of no surprise that the results displayed in Tab.2 are recover in Tab.3. The market risk

Model	$\tilde{\pi}_j(p)$	$mr_{dis}$	$mr_{ff}$	$mr$
1	0.0	6.9	-262.9	-256.0
2	71.6	-33.5	-294.0	-327.6
3	1305.1	-25.1	-1536.0	-1561.1
4	-571.7	14.2	301.5	315.7
5	15.8	46.2	-318.0	-271.9
6	-27.2	-48.7	-180.1	-228.8
7	37.6	-33.6	-260.1	-293.7
8	-250.3	-5.8	0.0	-5.8
9	-40.0	-216.0	0.0	-216.0
10	693.3	-949.3	0.0	-949.3
11	-24.8	-49.7	-181.5	-231.3
12	0.6	-71.7	-184.8	-256.6
13	1925.0	20920.6	-23101.6	-2181.0
14	-316.3	60.3	0.0	60.3
15	2633.6	-2889.6	0.0	-2889.6

Table 3: returns measures of market risk, model risk in distribution, model risk in functional form and total model risk for each of the models concerning the monetary value of the pension liabilities. Due to reasons of concealment the market risk is measured as the difference to the Wilkie model ( $M1$ ). The values are reported in  $1e+06€$ .

of  $M4$  undercuts the market risk of  $M1$  by 571.7 million €. On the other hand  $M4$  features the highest model risk describing again the above mentioned trade-off. With the multiplication factor being fixed in practice  $M4$  clearly induces the lowest capital reserves. As the next section shows this marks a very interesting result since  $M4$  is the model which is indeed chosen by application of an empirical model specification strategy.

## 6 Empirical Model Specification Strategy

The process of finding an appropriate model for the inflation scenarios marks a widely debated task among practitioners. In empirical work a specific model class is often chosen based on somewhat ideologic reasons and a suitable specification procedure is only very seldom carried out. Even if previous work attested a specific model to work very well for the economic variable at interest consulting a different data set might lead to a completely converse implication. That is why we propose a data driven approach concerning the process of model specification.

Our strategy consists of at most three steps and is given as follows. At first we try to find the best model in the class of linear time series. Once this model has been found, it is tested for remaining unspecified

nonlinearity. If the latter cannot be rejected the best linear model is tested against each of the nonlinear model classes given in Fig.4 being represented by their most general form. In other words we discriminate between the cases of short memory, long memory and spurious long memory. Whilst the decision between short and long memory is the decision between *ARFIMA* and *ARMA*, spurious long memory can be invoked by a nonlinear behavior of the process.

The selection of the most suited linear model might at first sight be thought of as an easy task since almost every conventional time series model is nested in 4.2. I.e. by selecting the lag orders in 4.2 first and estimating the corresponding parameters thereafter one might be able to impose zero restrictions on some of the parameters leading to sub models of the *ARFIMA* class. Hence a general-to-specific modeling procedure equivalent to the Box-Jenkins approach for *ARMA* models might be applied. There are however certain caveats in this argumentation.

Note that there are several ways to estimate the fractional differencing parameter in 4.2 which are based on the periodogram of the process. These include e.g. the *GPH* estimator of Geweke and Porter-Huwak [1983] or the Whittle estimator (cf. Robinson [1995]). The spectrum of a covariance stationary process  $\{y_t\}_{t=1}^T$  is given as

$$f(\lambda) = |1 - \exp(-i\lambda)|^{-2d} f^*(\lambda), \quad -\pi \leq \lambda \leq \pi, \quad |d| < 0.5 \quad (6.1)$$

with  $f^*(\lambda)$  representing the short-term correlation structure of the model and  $i = \sqrt{-1}$ . In practice 6.1 is approximated by the estimation function

$$\log I(\lambda_k) = c + dX_k + \varepsilon_k, \quad k = 1, \dots, m \quad (6.2)$$

for  $m \leq T/2$ ,  $X_k = -2\log(\sin \lambda_k)$ ,  $\lambda_k = 2\pi k/T$  and  $I(\lambda) = (2\pi T)^{-1} |\sum_{t=1}^T y_t \exp(it\lambda)|^2$  for the sample  $y_t$ ,  $t = 1, \dots, T$ . 6.2 is called the periodogram where  $c$  and  $d$  can be estimated via linear regression. However, as Hurvich et al. [1998] pointed out, the procedure of estimating  $d$  by 6.2 leads to a bias in case there are short-term correlations in the model, i.e. if  $f^*(\lambda)$  is not a constant. This induces that if 4.2 contains *ARMA* components no statements about the parameters' significance should be made for estimators based on 6.2.

Hence there are two possibilities for avoiding this shortcoming. Either one applies a different estimator not being based on the periodogram such as the nonparametric estimator proposed by Hurst [1951] or the maximum likelihood estimator of Beran [1995] determining all parameters simultaneously. Or the application of tests discriminating between short and long memory should be carried out. We decided for the second procedure as it is, to our knowledge, not assured that alternative estimators are robust against *ARMA* processes. Concretely we applied two tests in order to discriminate between short and long memory. Firstly we employed the test of Lo [1991] and secondly we applied the test of Davidson and Sibbertsen [2009].

Lo [1991] specifies a modified rescaled range estimator given by

$$\hat{Q}_T = \frac{\max_{0 < i \leq T} \{\sum_{t=1}^i (y_t - \bar{y})\} - \min_{0 < i \leq T} \{\sum_{t=1}^i (y_t - \bar{y})\}}{\hat{\sigma}_T} \quad \text{and} \quad (6.3)$$

$$\hat{\sigma}_T = T^{-1} \sum_{t=1}^T (y_t - \bar{y})^2 + 2T^{-1} \omega_j(q) \left( \sum_{t=j+1}^q (y_t - \bar{y})(y_{t-j} - \bar{y}) \right) \quad (6.4)$$

where  $\{y\}_{t=1}^T$  denotes the process of interest with mean  $\bar{y} = T^{-1} \sum_{t=1}^T y_t$  and  $\omega_j(q) = 1 - (j/(q+1))$  for  $q < T$ . Hence 6.3 can be interpreted as the range of partial sums of deviations of a time series from its mean, rescaled by its standard deviation. Note that 6.4 is the heteroscedasticity and autocorrelation consistent variance estimator with the weights  $\omega_j(q)$  being those suggested by Newey and West [1987]. Hence in case the process is short-range dependent  $\hat{\sigma}_T$  controls for the autocovariances making 6.3 able to discriminate between short-range and long-range dependence. As  $T^{-1/2} \hat{Q}_T$  is asymptotically distributed as the range of a standard brownian bridge under the Null of short-range dependence the latter can be tested against the alternative of long-range dependence.



The bias test of Davidson and Sibbertsen [2009] is based on 6.1 and tests  $H_0 : f^*(\lambda) = cons$  vs.  $H_1 : f^*(\lambda) \neq cons$ . The test statistic is given by

$$TS = \frac{\hat{d}_1 - \hat{d}_2}{SE(\hat{d}_1 - \hat{d}_2)} \quad (6.5)$$

where  $\hat{d}_1$  and  $\hat{d}_2$  denote alternative estimators of the fractional differencing parameter with  $SE(\cdot)$  being a suitable estimation for the difference's standard deviation being derived in Davidson and Sibbertsen [2009]. The authors further proved in their paper that  $TS \xrightarrow{d} N(0, 1)$  under certain conditions. Choosing  $\hat{d}_1$  to be the estimator regressing  $I(\lambda_k)$  onto  $(X_k, 1)$  in 6.2 whilst  $\hat{d}_2$  is derived if  $I(\lambda_k)$  is regressed onto  $(X_k, 1, h_1(\lambda_k), \dots, h_{pT}(\lambda_k))$ , where  $h_j(\lambda_k) = \cos(j\lambda_k)/\sqrt{\pi}$  is the  $j$ th order Fourier frequency, one can test the Null of an  $ARFIMA(0, d, 0)$  process against the alternative of an  $ARFIMA(p, d, q)$  process with either  $p > 0$  and/or  $q > 0$ . Note that 6.5 is a simple type of the Hausman [1978] test as under the Null  $\hat{d}_1$  is consistent and asymptotically efficient, but biased and inconsistent under the alternative, whereas  $\hat{d}_2$  is consistent under both hypotheses.

With a p-value being very close to zero the Lo test clearly rejects the Null of short-range dependence. The p-value of the bias test equals 0.045 indicating that there are  $ARMA$  components in the process at the 5% level. Having in mind the result of  $d$  being actually different from zero we can now be relatively sure that the presence of the fractional differencing parameter is due to long-range dependence and not spuriously caused by  $ARMA$  components although the latter are indeed present. Thus in the next step we estimated the parameters of the  $ARFIMA(p, d, q)$  model simultaneously by the method of Beran [1995] after selecting the lag orders via the Schwarz information criterion leading to the values reported in Tab.4 for M4.

Once a linear model has been specified and estimated it is tested against remaining nonlinearity. Note that there are several linearity tests in the literature (for an overview cf. Granger and Teräsvirta [1993]). We focused on testing against unspecified remaining nonlinearity as from a practical point of view it is not feasible to carry out different types of tests for every kind of nonlinear model. Thus we applied the popular test of Tsay [1986] performing considerably well in small samples as has been shown in simulation studies (cf. e.g. Tsay [1986] or Pena and Rodriguez [2005]). The test can be described as follows.

At first a linear model (in our case M4) is fitted to the time series and the residuals of the linear fit  $\hat{\epsilon}_t$  are computed. Secondly  $h = M(M + 1)/2$  proxy variables, where  $M$  stands for the autoregressive order of the process, are defined. The proxy variables are represented by  $z_t = vech(Y_t' Y_t)$  where  $Y_t = (y_{t-1}, \dots, y_{t-M})$  and  $vech(\cdot)$  denotes the column stacking operator using only those elements on or below the main diagonal of each column. Hence  $z_t$  consists of several squares and cross products of the series typifying the nonlinearity. Thirdly each of the proxy variables is regressed against  $Y_t$  and the  $h$  corresponding residuals are denoted by  $\hat{u}_t$ . Finally the model

$$\hat{\epsilon}_t = \xi \cdot \hat{u}_t + \eta_t \quad (6.6)$$

where  $\eta$  is white noise and  $\xi = (\xi_1, \dots, \xi_h)$  denotes a vector of coefficients. 6.6 is estimated by OLS and  $H_0 : \xi_1 = \dots = \xi_h = 0$  vs.  $H_1 : \xi_i \neq 0$ , for at least one  $i = 1, \dots, h$  is tested consulting a conventional F-test. Under the Null no remaining nonlinearity covered by the proxy variables can be detected. Having utilized the test we do not find remaining nonlinearity in M4 as the Tsay test reported a p-value of 0.75 for  $M = 3$ .<sup>8</sup>

## 7 Interest Rates

Until now the interest rate scenarios have been modeled exogenously, i.e. without regard of its reaction to the inflation scenarios. Furthermore its duration was given by theoretical considerations leading to the fact that

<sup>8</sup> $M$  was determined by information criteria. Choosing different orders did not alter the test's outcome.

model risk was solely determined by the inflation model. In this chapter the economic scenario generator is extended taking the impact of the interest rate's duration on model risk into account.

Note that there are mainly two approaches of modeling interest rates in the literature. The first marks the finance approach where affine no-arbitrage term structure models for the interest rate are utilized. The main disadvantage of this approach is that it is not economically motivated. That is little can be stated about the economic processes causing movements in the interest rate. This shortcoming can be avoided by modeling the interest rate by macro models being specified as the interest rate's reaction function to changes in economic variables such as e.g. inflation. Hence by noting that the benchmark economic scenario generators imply cascade structures with the inflation rate being the driving force, that is causality exclusively flowing from the inflation rate into interest rates without any feedback structure, modeling the interest rates via a macro model seems more appropriate for our analysis.

Recalling the shape of the pension function in Fig.3 it should be clear that model risk depends on the dynamics of the interest rate's forecast distribution. Hence the objective is to derive conclusions about the behavior of model risk in dependence on the forecast horizon of the interest rates. That is once the dynamics of the first two moments in dependence of  $h$  could be derived, statements about model risk can be deduced.

As a typical<sup>9</sup> macro model for interest rates we utilized the model of Clarida et al. [2000] being given by

$$y_t = (1 - \rho)\bar{x}_t + \rho y_{t-1} + \varepsilon_t, \quad \text{where} \quad (7.1)$$

$y_t$  denotes the funds rate at time  $t$ ,  $\rho$  describes a coefficient,  $\varepsilon_t$  is a normally distributed white noise term and  $\bar{x}_t$  is the desired funds rate. The latter however depends on the difference of the one quarter ahead expected inflation  $E(x_{t+1})$  and the inflation target  $\tilde{x}$  as well as on the expected output gap  $E(z_{t+1} - \tilde{z})$  measured as the deviation of log real gross domestic product from trend (cf. e.g. Taylor [1993]). With  $\Omega_t$  denoting the information set consisting of all relevant information up to and including time  $t$  the desired funds rate can be written as

$$\bar{x}_t = \tilde{y} + \phi_1 E(x_{t+1} - \tilde{x} | \Omega_t) + \phi_2 E(z_{t+1} - \tilde{z} | \Omega_t),$$

where  $\tilde{y}$  denotes the long term target for the funds rate. Hence 7.1 develops according to  $y_t = (1 - \rho)(\tilde{y} + \phi_1 E(x_{t+1} - \tilde{x} | \Omega_t) + \phi_2 E(z_{t+1} - \tilde{z} | \Omega_t)) + \rho y_{t-1} + \varepsilon_t$ .

It can now be shown (cf. A.1) that under non-consideration of the output gap's dynamics, i.e. for  $E(z_{T+k}) = \text{cons} \forall k \in \{1, \dots, H\}$  the dynamics of the expected  $h$ -step ahead forecast value of the funds rate are given by

$$\Delta E(y_{T+h}) = E(y_{T+h}) - E(y_{T+h-1}) \geq 0 \text{ for } \phi_0 + \underbrace{\phi_1 (E[x_{T+h} - \tilde{x} | \Omega_T])}_{\text{Deviation Inflation}} \geq \underbrace{E[y_{T+h-1} - \tilde{y} | \Omega_T]}_{\text{Deviation Interest}}, \quad (7.2)$$

where  $\phi_0$  denotes a constant term being defined in A.1. 7.2 can be interpreted as follows. If the inflation target deviation exceeds more than  $\phi_1^{-1}$  times the previous period's interest rate target deviation minus a constant, the expected value of the interest rate increases. If the inflation target deviation does not exceed more than  $\phi_1^{-1}$  times the previous period's interest rate target deviation minus a constant, the expected value of the interest rate decreases.

In order to gain more detailed results the inflation model should be specified and the dynamics of 7.2 should be further analyzed. If the inflation is again modeled with the benchmark inflation model of Wilkie [1995] being given by the simple AR(1) model  $x_t = \alpha_0 + \alpha_1 x_{t-1} + u_t$  the condition in 7.2 can be written as (cf. A.2)

$$\Delta E(y_{T+h} | \Omega_T) \geq 0 \quad \text{for} \quad \underbrace{\sum_{i=1}^{h-1} \left( \frac{\alpha_1}{\rho} \right)^{h-i}}_{:=g(h)} \geq \underbrace{\frac{y_T - \tilde{\phi}_0 - \phi_1 x_T}{\phi_1 (\alpha_0 + (\alpha_1 - 1)x_T)}}_{:=b} - 1, \quad (7.3)$$

<sup>9</sup>In his influential article Taylor [1993] developed the Taylor rule by modeling the funds rate as a function of inflation and the output gap. This model was refined by several authors like Goodhard [1992], Taylor [1999], Judd and Rudebusch [1998], Clarida et al. [2000] or Orphanides [2004], where the basic structure of Taylor's original model has been maintained.

where  $\tilde{\phi}_0$  denotes a constant being defined in A.2.

Obviously  $g(h)$  rises with  $h$  for  $a := \alpha_1/\rho > 0$ <sup>10</sup> whilst  $b$  remains unaltered. This means that one might find one  $0 \leq h^* < \infty$  after which the expected forecast value of the funds rate eventually increases. In other words for  $h \leq h^*$  the development of  $\Delta E(y_{T+h})$  depends on the initial conditions, i.e. on the size of the various parameters in 7.3 whereas for  $h > h^*$   $\Delta E(y_{T+h}) > 0$ .

In order to determine  $h^*$  the cases of  $0 < a < 1$  as well as  $a > 1$  ought to be discriminated (cf. A.3 in the following). In the former case  $\Delta E(y_{T+h}|\Omega_T)$  marks a convex function of  $h$  leading to the fact that  $\Delta E(y_{T+h}|\Omega_T) < 0$  for  $h < h_1^* = \ln(a + b(a-1))$  and  $b > 0$  whilst  $\Delta E(y_{T+h}|\Omega_T) > 0$  for  $h > h_1^*$ . However in case  $0 < a < 1$ ,  $\Delta E(y_{T+h}|\Omega_T)$  marks a concave function of  $h$  with  $\lim_{h \rightarrow \infty} \Delta E(y_{T+h}|\Omega_T) = a/(1-a)$ . Hence for  $b > a/(1-a)$  the expected forecast value of the funds rate permanently decreases. Note that there are two conditions in order for the latter condition to hold. First, the numerator and the denominator of the right-hand side term in 7.3 should feature the same sign and secondly  $|y_T - \tilde{\phi}_0 - \phi_1 x_T| > (1-a)^{-1} |\phi_1 (\alpha_0 + (\alpha_1 - 1)x_T)|$ . This behavior is illustrated in Fig.7

Note however that these statements do not suffice in order to derive implications concerning model risk as we

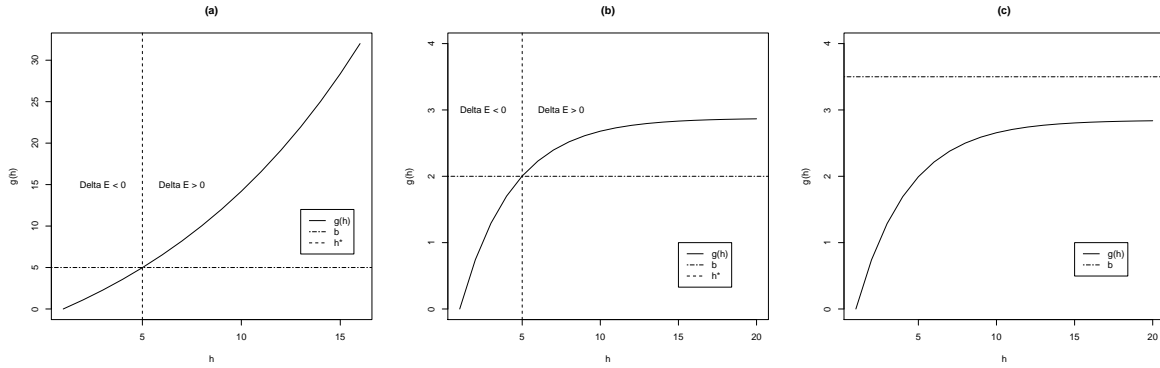


Figure 6: illustrates the behavior of  $\Delta E(y_{T+h})$  depending on  $g(h) := (a^h - a)/(1 - a)$  and the boundary  $b$ . In panel (a)  $a = 1.1$  and  $b = 5$  resulting in  $h_1^* = 5$ . Hence for  $h < 5$  the expected forecast value of the funds rate decreases whereas it increases for  $h > 5$ . The same dynamics can be detected in panel (b) with  $a = 0.75$ ,  $b = 2$  and  $h_2^* = 5$ . In panel (c) the case of a permanently decreasing expectancy is illustrated as again  $a = 0.75$  but  $b = 3.5 > a/(1-a)$ . In the latter case there is no intersection of  $g(h)$  and  $b$  and consequently  $b$  always exceeds  $(a^h - a)/(a - 1)$ .

additionally have to regard the forecast variance of the funds rate being given by  $V(y_{T+h}) = \sigma^2 \sum_{i=1}^h \rho^{2(h-i)}$  (cf. A.4). Straightforwardly

$$\Delta V(y_{T+h}) = \sigma^2 \rho^{2(h-1)}. \quad (7.4)$$

For  $\rho > 0$  as is the case in the existing literature (cf. Taylor [1999], Clarida et al. [2000] or Orphanides [2004]) the forecast variance marks a monotonically increasing (concave) function of  $h$ . In other words the interest rate's volatility increases with the forecast horizon, i.e. with the implied duration.

These results lead to the following statements. For  $a > 1$  model risk increases if firstly  $h < h_1^*$  and secondly  $b > 0$  as in this case the expected forecast value of the funds rate decreases while at the same time the forecast

<sup>10</sup>This condition should always hold as neither  $\alpha_1$  nor  $\rho$  are negative in empirical applications.

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variance increases with  $h$  indicating an inflation / interest rate combination in the area of the steeper ascend of the pension function. For  $h > h_1^*$  no distinct statement about the behavior of model risk concerning  $h$  can be made since the increase of the forecast variance might very well be overcompensated by an increasing expected value of the funds rate. For  $0 < a < 1$  the same conclusions hold with one exception. For  $b > a/(1-a)$ ,  $h \geq h_2^*$  can never be fulfilled indicating that model risk always increases with rising  $h$ .

These findings illustrate and elaborate three very important aspects of model risk. Firstly the forecast distribution of the funds rate is driven by the models being assumed for the interest rate as well as for the inflation. I.e. the model risk of the company model depends on the specification of the statistical models flowing into the ESG (cf. Fig.1). The second and third crucial components of model risk are given by the estimation procedure as well as the data. This becomes clear by recalling conditions 7.3 and 7.4 which determine the behavior of  $F(y_{T+h}|\Omega_T)$ . Both conditions merely include parameters to be estimated ( $\rho, \alpha_0, \alpha_1, \phi_1, \sigma$ ) and data values ( $x_T, y_T$ ). Whereas the estimated parameters depend on numerous considerations such as the estimation procedure, the choice of the sample period, as well as the data,  $x_T$  and  $y_T$  are exclusive components of the sample at hand. Note that these three aspects are linked very closely with each other as naturally the model specification determines possible estimation procedures which depend on the data while the latter again drive the model specification process. Hence when it comes to the analysis of model risk it seems reasonable to first consider several aspects of model risk such as specification, estimation and data rather than examine the components unconnectedly whilst secondly the interaction of those components should not be neglected.

## 8 Conclusion

In this paper we exemplified the interaction of an econometric model and the corresponding pension liabilities under the focus of induced model risk. For this purpose we modeled the inflation rate with 15 different economic scenario generators representing most of the conventional time series models in the literature. We then looked at the impact of model risk on capital reserves with the former functioning as a multiplication factor concerning market risk. We differentiated model risk into estimation risk, model risk functional form and model risk in distribution. The first striking result marks the fact that between the models the distribution of the inflation rate most substantially differs in its tails. Especially the class of *GARCH* models exhibits a high range as well as high kurtosis. This leads to a high estimation risk as well as considerable model risk in distribution for these kinds of models. The remaining models however feature rather decent (between 5 and 10 %) estimation risk. With model risk in distribution also being rather small total model risk is mainly caused by model risk in functional form. By using real insurance data we then determined the corresponding pension liabilities finding that induced model risk differs remarkably depending on the economic scenario generator that is applied. In general the model risk rises if the range and/or the kurtosis of the inflation scenarios increases. Concretely the discrepancy between the models might add up to several million € concerning capital reserves.

Furthermore we tried to objectively specify an inflation model by a data driven approach. By discriminating between short and long memory in the first place and thereafter testing for spurious long memory we predetermined the model class empirically. It was found that for the data set at hand long memory is indeed existent. Spurious long memory caused by nonlinearities could, however, not be detected. Hence the specification procedure signaled an *ARFIMA* process to be most appropriate. Remarkably the fitted process marks the model with the lowest induced model risk for the pension liabilities. Thus it can be stated that the task of model specification exhibits great influence on the related model risk.

Finally we examined the role of the interest rate model concerning model risk. By especially concentrating on the forecast horizon i.e. on the interest rate's duration being utilized in the pension function we find that there is a strong relation between model risk and the forecast horizon. Whether or not the former increases or decreases cannot be derived generally but mainly depends on the data.

Our analysis might be refined in two respects. First, we merely focused on cascade models as economic scenario generators. Here our work might easily be extended via utilization of (structural) multivariate models

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covering topics such as cointegration or causality. Furthermore consulting an alternative company model being dependent of more than two economic variables offers the analysis of the cascade structure of the Wilkie model in general and possible improvements thereof.

Secondly as was briefly mentioned in section 3, we did not deal with errors which might occur by leveraging the pension function. I.e. the selection of model points as well as the calibration of the pension function was neglected. Especially the first aspect is worth considering as it is still unclear which scenarios should be selected such that an appropriate fit of the pension function is achieved. Considering that an unrepresentative selection might lead to a bad fit misleading statements concerning the pension liabilities and the induced model risk might be concluded.

## A Proofs

### A.1 Deriving 7.2

Considering 7.1 the  $h$ -step forecast value of the interest rate is given as

$$y_{T+h} = (1-\rho) \sum_{k=1}^h \rho^{h-k} \bar{x}_{T+k} + \rho^h y_T + \sum_{k=1}^h \rho^{h-k} \varepsilon_{T+k}. \quad (\text{A.1})$$

Thus it follows that

$$\begin{aligned} E(y_{T+h}|\Omega_T) &= (1-\rho) \sum_{k=1}^h \rho^{h-k} E(\bar{x}_{T+k}|\Omega_T) + \rho^h y_T \\ &= \rho E(y_{T+h-1}|\Omega_T) + (1-\rho) E(\bar{x}_{T+h}|\Omega_T), \text{ since} \\ E(y_{T+h-1}|\Omega_T) &= (1-\rho) \left[ \rho^{h-2} E(\bar{x}_{T+1}) + \rho^{h-3} E(\bar{x}_{T+2}) + \dots + \rho E(\bar{x}_{T+h-2}) + E(\bar{x}_{T+h-1}) \right] + \rho^{h-1} y_T. \end{aligned}$$

Hence

$$\begin{aligned} \Delta E(y_{T+h}|\Omega_T) &= E(y_{T+h}|\Omega_T) - E(y_{T+h-1}|\Omega_T) \\ &= (\rho_1 - 1) E(y_{T+h-1}|\Omega_T) + (1-\rho) E(\bar{x}_{T+h}|\Omega_T) \\ &= (1-\rho) (E(\bar{x}_{T+h}|\Omega_T) - E(y_{T+h-1}|\Omega_T)) \\ \Rightarrow \Delta E(y_{T+h}|\Omega_T) &\geq 0 \text{ for } E(\bar{x}_{T+h}|\Omega_T) \geq E(y_{T+h-1}|\Omega_T). \end{aligned} \quad (\text{A.2})$$

Under non-consideration of the output gap's dynamics, i.e. for  $E(z_{T+k}) = \text{cons} \forall k \in \{1, \dots, H\}$

$$\begin{aligned} \bar{x}_{T+k} &= \tilde{y} + \phi_1 E(x_{T+k} - \tilde{x}|\Omega_T) + \phi_2 E(z_{T+k} - \tilde{z}|\Omega_T) \\ &= \underbrace{\phi_2 (E(z_{T+k}|\Omega) - \tilde{z})}_{:=\tilde{\phi}_0} + \phi_1 E(x_{T+k} - \tilde{x}|\Omega_T) + \tilde{y} \end{aligned}$$

which immediately results in 7.2.

### A.2 Deriving 7.3

Utilizing the Wilkie inflation model (Wilkie [1995]) being given by the simple AR(1) model  $x_t = \alpha_0 + \alpha_1 x_{t-1} + u_t$  leads to

$$E(x_{T+k}|\Omega_T) = \alpha_0 \sum_{i=1}^k \alpha_1^{k-i} + \alpha_1^k x_T. \quad (\text{A.3})$$

Recursively inserting into A.2 yields for

$h=2$ :

$$\begin{aligned} E(\bar{x}_{T+2}|\Omega_T) &= \underbrace{\tilde{y} - \tilde{x} + \phi_2 E(z_{T+2} - \tilde{z}|\Omega_T)}_{:=\tilde{\phi}_0} + \phi_1 E(x_{T+2}|\Omega_T) \\ &= \tilde{\phi}_0 + \phi_1 (\alpha_0 (1 + \alpha_1) + \alpha_1^2 x_T) \\ E(y_{T+1}|\Omega_T) &= \rho y_T + (1-\rho) E(\bar{x}_{T+1}|\Omega_T) = \rho y_T + (1-\rho) (\tilde{\phi}_0 + \phi_1 (\alpha_0 + \alpha_1 x_T)) \\ \Rightarrow \Delta E(y_{T+2}|\Omega_T) &= \underbrace{\phi_1 \alpha_1 (\alpha_0 + (\alpha_1 - 1) x_T)}_{=E(\bar{x}_{T+2}) - E(\bar{x}_{T+1})} + \rho (\tilde{\phi}_0 + \phi_1 (\alpha_0 + \alpha_1 x_T) - y_T) \\ &= E(\bar{x}_{T+2}) - E(\bar{x}_{T+1}) + \rho E(\bar{x}_{T+1}) + \rho y_T \\ &= \Delta E(\bar{x}_{T+2}) + \rho (E(\bar{x}_{T+1}) - y_T) \end{aligned}$$

$h = 3$ :

$$\begin{aligned}
E(\bar{x}_{T+3}|\Omega_T) &= \tilde{\phi}_0 + \phi_1(\alpha_0(1 + \alpha_1 + \alpha_1^2) + \alpha_1^3 x_T) \\
E(y_{T+2}|\Omega_T) &= \rho E(y_{T+1}) + (1 - \rho)E(\bar{x}_{T+2}|\Omega_T) \\
&= \rho^2 y_T + \rho(1 - \rho)(\tilde{\phi}_0 + \phi_1(\alpha_0 + \alpha_1 x_T)) + (1 - \rho)(\tilde{\phi}_0 + \phi_1(\alpha_0(1 + \alpha_1) + \alpha_1^2 x_T)) \\
\Rightarrow \Delta E(y_{T+3}|\Omega_T) &= \Delta E(\bar{x}_{T+3}) + \rho \Delta E(\bar{x}_{T+2}) + \rho^2 (E(\bar{x}_{T+1}) - y_T)
\end{aligned}$$

$\vdots$

$h = H$ :

$$\begin{aligned}
\Rightarrow \Delta E(y_{T+h}|\Omega_T) &= \Delta E(\bar{x}_{T+h}) + \rho \Delta E(\bar{x}_{T+h-1}) + \rho^2 \Delta E(\bar{x}_{T+h-2}) + \dots + \rho^{h-1} (E(\bar{x}_{T+1}) - y_T) \\
&= \sum_{i=1}^{h-1} \rho^{i-1} \Delta E(\bar{x}_{T+h-i+1}) + \rho^{h-1} (E(\bar{x}_{T+1}) - y_T), \quad \text{with} \\
\Delta E(\bar{x}_{T+h-i+1}) &= E(\bar{x}_{T+h-i+1}) - E(\bar{x}_{T+h-i}) \\
&= \tilde{\phi}_0 + \phi_1 E(x_{T+h-i+1}) - (\tilde{\phi}_0 + \phi_1 E(x_{T+h-i})) \\
&= \phi_1 (\alpha_0 \sum_{j=1}^{h-i+1} \alpha_1^{h-i+1-j} + \alpha_1^{h-i+1} x_T - (\alpha_0 \sum_{j=1}^{h-i} \alpha_1^{h-i-j} + \alpha_1^{h-i} x_T)) \\
&= \phi_1 (\alpha_0 \alpha_1^{h-i} + (\alpha_1^{h-i+1} - \alpha_1^{h-i}) x_T) \\
&= \phi_1 \alpha_1^{h-i} (\alpha_0 + (\alpha_1 - 1) x_T)
\end{aligned}$$

and thus

$$\Delta E(y_{T+h}|\Omega_T) = \phi_1 (\alpha_0 + (\alpha_1 - 1) x_T) \sum_{i=1}^{h-1} \rho^{i-1} \alpha_1^{h-i} + \rho^{h-1} (\tilde{\phi}_0 + \phi_1 (\alpha_0 + \alpha_1 x_T) - y_T).$$

Hence it follows that

$$\begin{aligned}
\Delta E(y_{T+h}|\Omega_T) &\geq 0 \quad \text{for} \\
\sum_{i=1}^{h-1} \rho^{i-1} \alpha_1^{h-i} &\geq -\rho^{h-1} (\tilde{\phi}_0 + \phi_1 (\alpha_0 + \alpha_1 x_T) - y_T) / (\phi_1 (\alpha_0 + (\alpha_1 - 1) x_T)) \\
\Leftrightarrow \sum_{i=1}^{h-1} \rho^{i-h} \alpha_1^{h-i} = \sum_{i=1}^{h-1} \left( \frac{\alpha_1}{\rho} \right)^{h-i} &\geq \frac{y_T - \tilde{\phi}_0 - \phi_1 (\alpha_0 + \alpha_1 x_T)}{\phi_1 (\alpha_0 + (\alpha_1 - 1) x_T)} = \frac{y_T - \tilde{\phi}_0 - \phi_1 x_T}{\phi_1 (\alpha_0 + (\alpha_1 - 1) x_T)} - 1
\end{aligned}$$

### A.3 Deriving $h^*$

Defining  $a := \alpha_1/\rho$  and  $b := (y_T - \tilde{\phi}_0 - \phi_1 x_T) / (\phi_1 (\alpha_0 + (\alpha_1 - 1) x_T)) - 1$  and determining

$$\sum_{i=1}^{h-1} a^{h-i} = \frac{a^h - a}{a - 1}$$

with  $a \neq 1$  yields for

(i)  $a > 1$ :

$$\begin{aligned}
\Delta E(y_{T+h}|\Omega_T) \geq 0 &\Leftrightarrow \frac{a^h - a}{a - 1} \geq b \\
&\Leftrightarrow h \geq \frac{\ln(a + b(a - 1))}{\ln a} = h_1^* \quad \text{where } b \stackrel{!}{>} \frac{-a}{a - 1}
\end{aligned}$$



and (ii)  $0 < a < 1$ :

$$\begin{aligned} \Delta E(y_{T+h}|\Omega_T) \geq 0 &\Leftrightarrow \frac{a^h - a}{a - 1} \geq b \\ &\Leftrightarrow h \leq \frac{\ln(a + b(a - 1))}{\ln a} = h_2^* \quad \text{where } b < \frac{-a}{a - 1} \end{aligned}$$

Note that for the unlikely case of  $a = 1$ ,  $\Delta E(y_{T+h}|\Omega_T) \geq 0$  for  $h \geq b + 1$ .

#### A.4 Forecast Variance

By consideration of A.1

$$V(y_{T+h}) = E[(y_{T+h} - E(y_{T+h}))(y_{T+h} - E(y_{T+h}))] = E\left[\left(\sum_{k=1}^h \rho_1^{h-k} \varepsilon_{T+k}\right)\left(\sum_{k=1}^h \rho_1^{h-k} \varepsilon_{T+k}\right)\right].$$

With  $\varepsilon$  being white noise the forecast variance is given as

$$\begin{aligned} V(y_{T+h}) &= E\left[(\rho_1^{h-1} \varepsilon_{T+1} + \rho_1^{h-2} \varepsilon_{T+2} + \dots + \rho_1 \varepsilon_{T+h-1} + \varepsilon_{T+h})^2\right] \\ &= \sigma^2 \left(\rho_1^{2(h-1)} + \rho_1^{2(h-2)} + \dots + \rho_1^2 + 1\right) \\ &= \sigma^2 \sum_{i=1}^h \rho_1^{2(h-i)} \end{aligned}$$

resulting in 7.4 since

$$\Delta V(y_{T+h}) = V(y_{T+h}) - V(y_{T+h-1}) = \sigma^2 \rho_1^{2(h-1)}.$$

## B Tables

### B.1 Tab.4

Statistic	Model														
	M1	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12	M13	M14	M15
Minimum	-0.18	-0.21	-0.33	-0.09	-0.21	-0.16	-0.14	-0.16	-0.25	-1.09	-0.19	-0.20	-1.70	-0.15	-1.24
1%-Quantile	-0.10	-0.11	-0.20	-0.04	-0.13	-0.10	-0.07	-0.08	-0.10	-0.18	-0.10	-0.11	-0.22	-0.08	-0.25
5%-Quantile	-0.07	-0.08	-0.13	-0.03	-0.09	-0.07	-0.04	-0.05	-0.06	-0.08	-0.06	-0.07	-0.11	-0.05	-0.12
1st Quartile	-0.01	-0.02	-0.04	0.00	-0.03	-0.02	0.00	-0.01	-0.01	-0.01	-0.01	-0.01	-0.02	-0.01	-0.03
Median	0.02	0.02	0.02	0.02	0.01	0.02	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
Mean	0.02	0.02	0.02	0.02	0.01	0.02	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
3rd Quartile	0.06	0.06	0.09	0.04	0.05	0.05	0.07	0.05	0.05	0.05	0.06	0.06	0.06	0.05	0.06
95%-Quantile	0.11	0.12	0.18	0.07	0.11	0.10	0.11	0.09	0.10	0.12	0.11	0.11	0.15	0.09	0.17
99%-Quantile	0.15	0.15	0.24	0.09	0.15	0.14	0.14	0.12	0.14	0.21	0.14	0.15	0.26	0.12	0.36
Maximum	0.22	0.25	0.52	0.12	0.23	0.21	0.22	0.18	0.35	0.84	0.20	0.23	0.74	0.20	2.95
1st Moment	0.02	0.02	0.02	0.02	0.01	0.02	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
2nd Moment	0.05	0.06	0.10	0.03	0.06	0.05	0.05	0.04	0.05	0.07	0.05	0.05	0.09	0.04	0.11
3rd Moment	-0.02	-0.03	0.02	0.01	-0.03	-0.05	0.00	-0.03	0.07	-0.33	-0.03	-0.01	-1.4	-0.05	3.42
4th Moment	-0.04	-0.01	0.00	-0.01	-0.02	-0.05	-0.05	0.02	1.14	18.5	0.04	0.15	27.2	0.06	78.53

Table 4: gives some descriptive statistics of the  $N = 10,000$  simulated inflation path according to the respective model. The 4th moment corresponds to excess kurtosis compared to the normal distribution.

## B.2 Tab.5

Statistic	Model														
	M1	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12	M13	M14	M15
Minimum	0	-15.2	-17.3	185.8	-14.3	12.4	54.0	9.8	-17.0	-17.3	-6.4	-11.0	-17.3	27.6	-17.3
1%-Quantile	0	-27.0	-131.1	202.7	-54.2	-3.2	89.0	73.9	17.3	-89.5	4.2	-3.1	-119.4	64.0	-125.3
5%-Quantile	0	-35.2	-178.9	166.1	-71.2	-9.1	84.1	64.5	38.9	-36.4	7.6	-7.2	-111.1	63.5	-117.0
1st Quartile	0	-23.0	-137.5	86.6	-63.2	-17.7	97.6	38.2	29.4	9.5	9.8	-0.1	-35.6	35.7	-49.6
Median	0	-16.6	-8.5	-10.6	-54.6	-22.7	83.6	-5.1	-16.0	-6.8	-6.0	-8.2	-11.2	-9.5	-34.3
Mean	0	-7.3	100.3	-42.7	-44.9	-32.5	7.1	-21.5	-19.5	28.6	-6.5	-9.7	60.6	-26.3	168.5
3rd Quartile	0	3.9	215.5	-135.6	-36.3	-42.9	68.5	-59.9	-63.4	-34.6	-15.4	-17.8	39.5	-68.8	14.7
95%-Quantile	0	25.8	757.7	-376.9	5.1	-89.2	25.3	-178.5	-112.6	117.4	-44.6	-24.8	377.3	-185.9	673.5
99%-Quantile	0	84.2	1186.4	-563.9	72.4	-92.0	2.6	-266.7	-68.2	755.9	-48.8	1.1	1411.9	-294.3	3358.4
Maximum	0	693.0	6148.0	-1011.0	129.0	-39.0	239.0	-52.02	2039.0	19007.0	-34.0	267.0	14417.0	-152.0	230937.0
1st Moment	0	-7.3	100.3	-42.7	-44.9	-32.5	78.0	-21.5	-19.5	28.6	-6.5	-9.7	60.6	-26.3	168.5
2nd Moment	0	21.9	310.7	-169.6	25.9	-22.1	-20.1	-77.6	-37.1	247.0	-14.8	-4.9	305.2	-79.8	2596.1
3rd Moment	0	0.1	0.6	-0.3	0.0	0.9	-0.1	-0.2	0.5	10.1	0.0	0.1	4.6	-0.2	56.8
4th Moment	0	0.3	3.1	-0.5	1.2	0.0	-0.1	-0.3	5.3	262.5	0.0	0.5	69.4	-0.2	4144.4
VaR 0.995	0	77.0	1405.5	-668.8	119.7	-80.0	9.0	-304.0	-3.2	1407.5	-46.1	36.7	2201.3	-334.9	5048.6
TVaR 0.995	0	121.4	1728.7	-768.5	128.2	-115.0	-8.5	-374.9	151.3	3758.3	-56.0	64.9	3892.3	-381.9	17591.1

Table 5: gives some descriptive statistics of the distribution of the  $N = 10,000$  pension liabilities measured as the difference to the Wilkie model (M1). The values are reported in  $1e+06\text{€}$ . The 4th moment corresponds to excess kurtosis compared to the normal distribution.

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