Validate Correlation of an ESG: Treasury Yields across Economies

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Abstract

Within an internal model the Economic Scenario Generator (ESG) is an important component. In order to get a regulatory approval of an internal model it is required that the implemented models (must be) passed a rigorous validation process, see Ceiops [2009].

In this paper we focus on the particular problem to judge the contribution of correlations between interest rate risks across countries in the ESG. To that end we apply two strategies: an analytical and a statistical one. The analytical approach yields necessary conditions in terms of upper and lower bounds for correlations within the chosen model. A system of stochastic differential equations is used to describe several economies simultaneously. In this framework we derive a lower and upper bound of the correlation of the treasury yields between two economies by solving the associated ordinary differential inequalities.

In order to deepen our understanding about the correlation structure we consider three modeling types of correlations of historical datasets. We first derive the realized correlations as outlined by Andersen et al. [2003] for the historical treasury yields of two economies. Furthermore we include Engle's parsimonious multivariate GARCH models – known as Dynamical Conditional Correlation (DCC) model, see Engle [2009] – and we derive conditional correlations out of our ESG. We then exploit a nice relationship outlined by Andersen et al. [2003], which relates the realized correlation and conditional correlations in oder to compare the three model by their ability to capture the stylized facts of the underlying processes. In this respect the long memory of the correlation processes is of particular importance. We give a series of statistical analysis that highlight the adequacy of the model.

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1 Introduction

Under Solvency II, the internal model approach is chosen by many insurance companies. Within an internal model the Economic Scenario Generator (ESG) is of particular importance. Typically insurance undertakings use ESGs provided by external vendors. In order to get a regulatory approval of an internal model it is required that the implemented models (must be) passed a rigorous validation process, see Ceiops [2009]. This is in particular the case for models delivered by third parties, because the use of black boxes is not in line with the spirit of Solvency II. This situation – using an externally provided ESG within an internal model – is the starting point and motivation of the following exposition.

In this paper we focus on the particular, however important problem to judge the contribution of correlations between interest rate risks across countries to the diversification or concentration. To that end we apply two strategies: an analytical and a statistical one. The analytical approach yields necessary conditions in terms of upper and lower bounds for correlations within the chosen model. For a fixed economy e the short rate is defined by a three factor extended Cox-Ingersoll-Ross(CIR3++) models, see e.g. Chen and Scott [2002]. A system of stochastic differential equations is used to describe several economies simultaneously. In order to ease the exposition we consider the case of two economies. In this framework we derive a lower and upper bound of the correlation of the treasury yields between these two economies (i.e. for two correlated CIR process $X_i^{e_1},\ X_i^{e_2}$ of two economies $e_1,\ e_2$)

$$\int_0^t \mathbb{E}[X_i^{e_1} X_i^{e_2}]\mathrm{d}s \leq \int_0^t \mathbb{E}[\sqrt{X_i^{e_1} X_i^{e_2}}]\mathrm{d}s \leq \int_0^t \sqrt{\mathbb{E}[X_i^{e_1} X_i^{e_2}]}\mathrm{d}s \tag{1.1}$$

by solving associated ordinary differential inequalities in line with (1.1) (see theorem 2.1).

In order to deepen our understanding about the correlation structure we consider three modeling types of conditional correlations. We start with the history of treasury yields and derive thereof realized correlations as outlined by Andersen et al. [2003]. Furthermore we include Engle's parsimonious multivariate GARCH models – known as Dynamical Conditional Correlation (DCC) model, see Engle [2009] – and we derive conditional correlations out of our ESG. Now, we exploit a nice relationship outlined by Andersen et al. [2003], which relates the realized correlation and conditional correlations, especially the DCC, in oder to compare the three model by their ability to capture the stylized facts of the underlying processes. In this respect the long memory of the correlation processes is of particular importance. We give a series of statistical analysis that highlight the adequacy of the model.

The rest of paper is organized as following. In section 2 we validate the correlation of treasury yields by testing its empirical values based on the scenarios generated by the ESG. For this purpose we first derive a upper and a lower bound for the correlation of the treasury yield model of the ESG. Then we compare the empirical value of the correlation implied by the ESG with its boundary. This is illustrated for the case of two economies of Germany and U.S.A. In the section section 3 we validate the dynamic correlation of the treasury yields across economies. After giving the definition of dynamic (conditional) correlation in line with Engle [2009] and the realized correlation, we introduce a fundamental relationship between them found by Andersen et al. [2003]. Focusing on one important stylized facts of the conditional correlations (long memory), we compare the performance of the ESG with two alternative models, the HAR-model for realized correlation and the DCC-model on the historical treasury yields of Germany and U.S.A.. In order to calculate the historical dynamic conditional correlation of the ESG-model we apply two approximations: one is our upper bound derived in the first section, the other one is the so-called *Gaussian-mapping*. In the last section 4 we give a brief outlook for possible future works.

2 Validate the correlation of the scenarios for treasury yields

In the first part of the paper we validate the correlations of the scenarios for treasury yields across the economies. We first provide some theoretical results about the correlations based on the interest rate models. Then we perform some empirical tests on the scenarios generated by the ESG against our theoretical analysis.

2.1 Some theoretical results

Within this ESG the risk free term structure under the real world measure is modeled through the three-factor extended Cox-Ingersoll-Ross (CIR3++) model. The zero bond curves and coupon curves are derived from this short rate model.

For a fixed economy e let $X^e(t) = (X_i^e(t))_{i=1}^3 \in \mathbb{R}^3$ be the state variable given by

$$dX_i^e(t) = \kappa_i^e(\theta_i^e - X_i^e(t))dt + \sigma_i^e\sqrt{X_i^e(t)}dW_i^e(t), \tag{2.1}$$

where $\mathbf{W}^e(t) = (W_i^e(t))_{i=1}^3$ is a three dimensional standard Brownian motion. Without loss of the generality we have omitted the *market price of risks* which are constant and can be encompassed into the κ and θ in a proper way during the computing [1].

The short rate $r^e(t)$ is given by

$$r^e(t) = \delta^e(t) + \sum_{i=1}^{3} X_i^e(t), \tag{2.2}$$

where $\delta : \mathbb{R}_+ \to \mathbb{R}$ is a piecewise linear function.

Without loss of generality we assume that $\delta(t) = 0$ for all t for the theoretical analysis.

Since CIR3++ model belongs to the *affine term structure* model, the bond price process can be represented as

$$P^e(t,T) = \left(A^e(t,T)\exp\left(-\mathbf{B}^e(t,T)'X^e(t)\right)\right), \tag{2.3}$$

where $A^e(t,T)$ and $\mathbf{B}^e(t,T) = (B_i^e(t,T))_{i=1}^3$ follow certain ordinary Differential equations, see, e.g. Chen and Scott [2002] for the CIR3 model and Brigo and Mercurio [2006] for the CIR2++ model. Analogous we can derive that

$$\begin{aligned} A^e(t,T) &= \prod_{i=1}^{3}\left[\frac{2h_i^e\exp\{(k_i^e+h_i^e)(T-t)/2\}}{2h^e+(k_i^e+h_i^e)(\exp\{(T-t)h_i^e\}-1)}\right]^{2k_i^e\theta_i^e/(\sigma_i^e)^2}, \\ B_i^e(t,T) &= \frac{2(\exp\{(T-t)h_i^e\}-1)}{2h^e+(k_i^e+h_i^e)(\exp\{(T-t)h_i^e\}-1)}, \end{aligned} \tag{2.4}$$

where $h_i^e := \sqrt{(\kappa_i^e)^2 + 2(\sigma_i^e)^2}$.

Let $R^e(t,T) := -\frac{\log P^e(t,T)}{T-t}$ denote the yield curves of $P^e(t,T)$. Thus we have

$$R^e(t,T) = -\frac{\log A^e(t,T) - \sum_{i=1}^3 B_i^e(t,T)X_i^e(t))}{T-t}. \tag{2.5}$$

Let us assume that for arbitrary two economies $e1$ and $e2$ their yield curves are correlated through the correlated underlying Brownian Motions $\mathbf{W}^{e1}$ and W^{e2} as following:

$$\begin{aligned} dW_i^{e1}(t)dW_i^{e2}(t) &= \rho_i^{e1,e2}dt, \quad i = 1,2,3, \\ dW_i^{e1}(t)dW_j^{e2}(t) &= 0, \quad i \neq j, \end{aligned}$$

where $-1 \leq \rho_i^{e1,e2} \leq 1$. And thus we have

$$\mathbb{C}ov\left[X_i^{e1}(t), X_j^{e2}(t)\right] = 0, \quad i \neq j.$$

[1] That is, in the real-world dynamic of the bond price (2.3) the affine terms A and B depend on the risk-neutral parameters and the dynamics of state variable $X(t)$ depend on the real-world parameters.

Proposition 2.1. ***(Correlation of yield curves between two economies)***
Let $R^{e1}(t,T)$,$R^{e2}(t,S)$ be the yield curves of two arbitrary economies $e1$ and $e2$ with maturity T and S respectively. The correlation between these two yield curves is given by

$$\begin{aligned}
&\mathbb{C}orr[R^{e1}(t,T), R^{e2}(t,S)] \\
&= \frac{\mathbb{C}ov\left[R^{e1}(t,T), R^{e2}(t,S)\right]}{\sqrt{\mathbb{V}ar\left[R^{e1}(t,T)\right]\mathbb{V}ar\left[R^{e2}(t,S)\right]}} \\
&= \frac{\mathbb{C}ov\left[\sum_{i=1}^{3} B_i^{e1}(t,T)X_i^{e1}(t), \sum_{i=1}^{3} B_i^{e2}(t,S)X_i^{e2}(t)\right]}{\sqrt{\mathbb{V}ar\left[\sum_{i=1}^{3} B_i^{e1}(t,T)X_i^{e1}(t)\right]\mathbb{V}ar\left[\sum_{i=1}^{3} B_i^{e2}(t,S)X_i^{e2}(t)\right]}} \\
&= \frac{\sum_{i=1}^{3} B_i^{e1}(t,T)B_i^{e2}(t,S)\mathbb{C}ov\left[X_i^{e1}(t), X_i^{e2}(t)\right]}{\sqrt{\sum_{i=1}^{3}(B_i^{e1}(t,T))^2 \cdot \mathbb{V}ar\left[X_i^{e1}(t)\right]\sum_{i=1}^{3}(B_i^{e2}(t,S))^2 \cdot \mathbb{V}ar\left[X_i^{e2}(t)\right]}} \\
&= \frac{\sum_{i=1}^{3} B_i^{e1}(t,T)B_i^{e2}(t,S)\big(\mathbb{E}[X_i^{e1}(t)X_i^{e2}(t)] - \mathbb{E}[X_i^{e1}(t)]\mathbb{E}[X_i^{e2}(t)]\big)}{\sqrt{\sum_{i=1}^{3}(B_i^{e1}(t,T))^2 \cdot \mathbb{V}ar\left[X_i^{e1}(t)\right]\sum_{i=1}^{3}(B_i^{e2}(t,S))^2 \cdot \mathbb{V}ar\left[X_i^{e2}(t)\right]}},
\end{aligned} \tag{2.6}$$

where in the second to last step we use the fact that only the $(W_i^{e1}(t), W_i^{e2}(t))$ are correlated.

It is sufficient to calculate the covariance of the related state variables $X_i^{e1}, X_i^{e2}, \quad i = 1,2,3$. The main task now turns out to be calculating the following terms

$$\mathbb{E}[X_i^{e1}(t)X_i^{e2}(t)].$$

For clarity of the formulation we denote from now on X, Y two arbitrary correlated CIR processes given by

$$\mathrm{d}X(t) = \kappa(\theta - X(t))\mathrm{d}t + \sigma\sqrt{X(t)}\mathrm{d}W(t), \tag{2.7}$$

$$\mathrm{d}Y(t) = \bar{\kappa}(\bar{\theta} - Y(t))\mathrm{d}t + \bar{\sigma}\sqrt{Y(t)}\mathrm{d}\bar{W}(t), \tag{2.8}$$

with $\mathrm{d}W(t)\mathrm{d}\bar{W}(t) = \rho dt$ being two correlated Brownian Motions.

The mean and the variance of $X(t)$ conditional on filtration $\mathcal{F}_s$ at time s are given by

$$\begin{aligned}
\mathbb{E}[X(t)|\mathcal{F}_s] &= X(s)\exp\{-\kappa(t-s)\} + \theta\left(1 - \exp\{-\kappa(t-s)\}\right), \\
\mathbb{V}ar[X(t)|\mathcal{F}_s] &= X(s)\frac{\sigma^2}{\kappa}\left(\exp\{-\kappa(t-s)\} - \exp\{-2\kappa(t-s)\}\right) + \theta\frac{\sigma^2}{2\kappa}\left(1 - \exp\{-\kappa(t-s)\}\right)^2.
\end{aligned} \tag{2.9}$$

Theorem 2.1. ***(Correlation boundary for correlated CIR processes)***
Let us define $h(t) = \mathbb{E}\left[X(t)Y(t)\right]$ for all $t \leq 0$.
Let us define

$$\Delta(t) := \bar{\kappa}\bar{\theta}\mathbb{E}[X(t)] + \kappa\theta\mathbb{E}[Y(t)], \tag{2.10}$$

$$m := -(\kappa + \bar{\kappa}), \tag{2.11}$$

$$n := \sigma\bar{\sigma}\rho. \tag{2.12}$$

Let f, g be the solutions of the following two ODEs

$$\Delta(t) + (m+n)f(t) = f'(t), \tag{2.13}$$

$$\Delta(t) + mg(t) + n\sqrt{g(t)} = g'(t), \tag{2.14}$$

where $f(0) = g(0) = h(0)$.
Then the term $h(t)$ satisfies following inequality

$$f(t) \leq h(t) \leq g(t). \tag{2.15}$$

Proof. We first apply Ito's formula on $X(t)Y(t)$ and get

$$X(t)Y(t) - X(0)Y(0) = \int_0^t \left(\bar{\kappa}\bar{\theta}X(s) + \kappa\theta Y(s)\right) \mathrm{d}s \tag{2.16}$$

$$+ \int_0^t mX(s)Y(s)\mathrm{d}s + \int_0^t n\sqrt{X(s)Y(s)}\mathrm{d}s \tag{2.17}$$

$$+ \int_0^t (\cdot)\mathrm{d}W(s) + \int_0^t (\cdot)\mathrm{d}\bar{W}(s). \tag{2.18}$$

Now we take the expectation on both sides of the above equation. Applying the Fubini theorem and using the fact that the expectation of Ito integral w.r.t. Brownian Motion is zero, we get

$$\mathbb{E}[X(t)Y(t)] - X(0)Y(0) = \int_0^t \Delta(s)\mathrm{d}s + \int_0^t m\mathbb{E}[X(s)Y(s)]\mathrm{d}s + \int_0^t n\mathbb{E}[\sqrt{X(s)Y(s)}]\mathrm{d}s. \tag{2.19}$$

Note that under our modeling settlement $0 \leq X(t)Y(t) \leq 1$ and that $\sqrt{\cdot}$ is a concave function. By applying the fact $X(t)Y(t) \leq \sqrt{X(t)Y(t)}$ and Jensen's inequality we have

$$\int_0^t n\mathbb{E}[X(s)Y(s)]\mathrm{d}s \leq \int_0^t n\mathbb{E}[\sqrt{X(s)Y(s)}]\mathrm{d}s \leq \int_0^t n\sqrt{\mathbb{E}[X(s)Y(s)]}\mathrm{d}s. \tag{2.20}$$

Substitute this inequality into 2.19 we have following two differential inequalities

$$\Delta(t) + (m+n)h(t) \leq h'(t) \leq \Delta(t) + mh(t) + n\sqrt{h(t)}. \tag{2.21}$$

Let f, g be the solutions of the following two ODEs

$$\Delta(t) + (m+n)f(t) = f'(t), \tag{2.22}$$

$$\Delta(t) + mg(t) + n\sqrt{g(t)} = g'(t), \tag{2.23}$$

with $f(0) = g(0) = h(0)$. Furthermore, let $U_1(t, f) := -\Delta(t) + (m+n)f$ and $U_2(t, g) := \Delta(t) + mg + n\sqrt{g}$. We see that both U_1 and U_2 are continuously differentiable on interval $(0, \infty) \times (0, \infty)$, therefore they are local Lipschitz.

Now by applying the lemma 4.1 in appendix we get the desired results. □

The equation of (2.13) is a first order linear ODE and its solution is given by

$$f(t) = f(0)\exp\{(m+n)t\} + \exp\{(m+n)t\}\int_0^t \Delta(s)\exp\{-(m+n)s\}\mathrm{d}s, \tag{2.24}$$

where $f(0) = h(0)$. (see, e.g. Walter [1998] page 28).

The equation of (2.14) is a so called *nonlinear nonautonomous first order differential equation*. If $\Delta(t) = 0$ it becomes a Bernoulli's differential equation (see, e.g. Walter [1998] page 29). So we call the equation (2.14) *inhomogeneous Bernoulli's equation*. Since to our best knowledge no analytical solution to such an equation is available, we will solve it numerically.

Example 2.1. ***(Case of two correlated Vasicek processes)***
Let X, Y be two correlated Vasicek processes which are given by

$$\mathrm{d}X(t) = \kappa(\theta - X(t))\mathrm{d}t + \sigma\mathrm{d}W(t),$$
$$\mathrm{d}Y(t) = \bar{\kappa}(\bar{\theta} - Y(t))\mathrm{d}t + \bar{\sigma}\mathrm{d}\bar{W}(t),$$

with $\mathrm{d}W(t)\mathrm{d}\bar{W}(t) = \rho dt$ being two correlated Brownian Motions.

Let $h(t) := \mathbb{E}[X(t)Y(t)]$ and Δ, m, n be defined as in the Theorem 2.1 for X, Y. Then similar to the Theorem 2.1 we can derive that h satisfies following ODE

$$\Delta(t) + n + mh(t) = h'(t),$$

which has a solution

$$h(t) = h(0)\exp\{mt\} + \exp\{mt\}\int_0^t (\Delta(s) + n)\exp\{-ms\}\mathrm{d}s.$$

We see that for the case of correlated Vasicek processed we can derive the correlation analytically.

2.2 Empirical tests: case of Germany and U.S.A.

As an example we analysis the ESG scenarios of the economies Germany (DE) and U.S.A.(US). The simulation starts at $30.09.2009$. The simulated horizon is up to 50 years in the future and the simulated treasury yields have terms (time to maturities) from 0.25 to 35 years. The number of the simulated paths is 10.000. Precisely, we choose following settings for both of DE and US:

Table 1: Simulated Horizon of the Scenarios of Treasury Yields: DE and US

Risk Horizon(year)	1.0	2.0	3.0	4.0	5.0	10.0	15.0	20.0	25.0	30.0

Table 2: Terms of Treasury Yields in the Scenarios: DE and US

Pivot Maturity: DE	3M	5Y	10Y
Pivot Maturity: US	3M	5Y	10Y

Based on the scenarios generated by our ESG, we first calculate the empirical correlation for the yields curves of Germany and U.S.A. which is the statistic estimate of formula (2.6) (see figure 1 and table of 3 for its values). Then by using the parameters of the ESG and applying our approximations in theorem (2.1) we derive its upper bounds and lower bounds (see figures of 2 and table of 4 for upper bounds and table of 5 for lower bounds respectively)[2].

From this empirical test we can see that the empirical correlation for (2.6) lies within its upper and lower bounds. Furthermore, by observing the difference between the upper bounds and the empirical correlation (see figure 3) we see that all of the empirical correlations are lower (but quite close) to their theoretical upper bounds. The empirical correlations are obviously larger than their lower bounds (see table 5).

Therefore we can conclude that based on our approximation the correlation of the treasury yields between Germany and U.S.A. of our ESG is plausible. In the next section we will use this upper bound approximation to validate the dynamical correlations of the ESG.

[2]Here we use the open source R-package *deSolve*Soetaert et al. [2010] to solve the ordinary differential equations involved in our approximation.

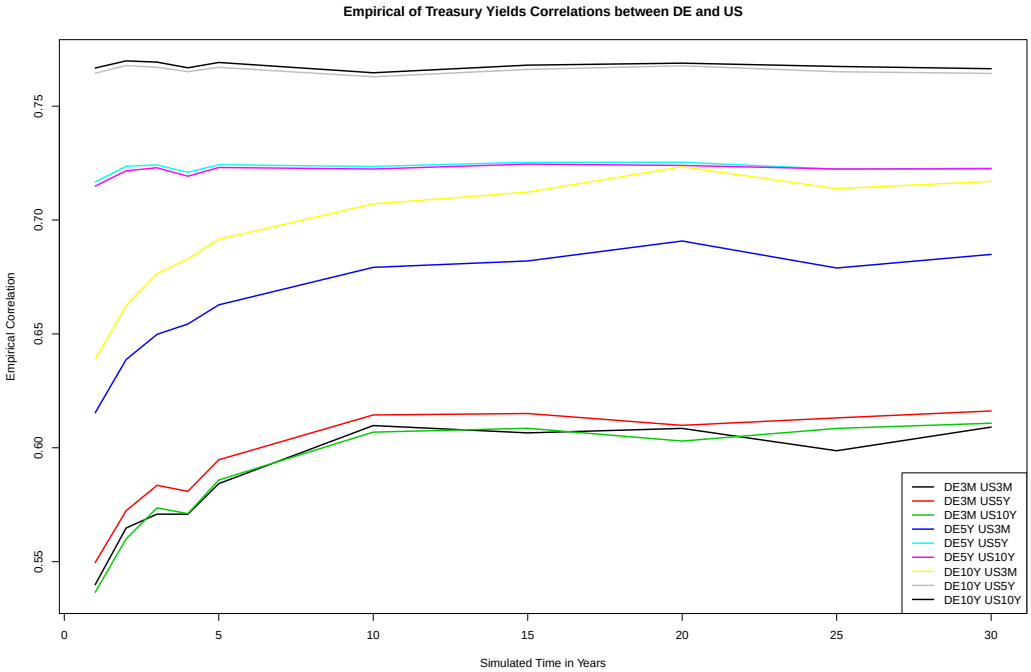


Figure 1: Empirical correlation of the Treasury Yields: DE and US

Table 3: Empirical Correlation of Treasury Yields: DE and US

	1	2	3	4	5	10	15	20	25	30
DE3M US3M	0.54	0.56	0.57	0.57	0.58	0.61	0.61	0.61	0.60	0.61
DE3M US5Y	0.55	0.57	0.58	0.58	0.59	0.61	0.62	0.61	0.61	0.62
DE3M US10Y	0.54	0.56	0.57	0.57	0.59	0.61	0.61	0.60	0.61	0.61
DE5Y US3M	0.62	0.64	0.65	0.65	0.66	0.68	0.68	0.69	0.68	0.68
DE5Y US5Y	0.72	0.72	0.72	0.72	0.72	0.72	0.73	0.73	0.72	0.72
DE5Y US10Y	0.71	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72
DE10Y US3M	0.64	0.66	0.68	0.68	0.69	0.71	0.71	0.72	0.71	0.72
DE10Y US5Y	0.76	0.77	0.77	0.77	0.77	0.76	0.77	0.77	0.77	0.76
DE10Y US10Y	0.77	0.77	0.77	0.77	0.77	0.76	0.77	0.77	0.77	0.77

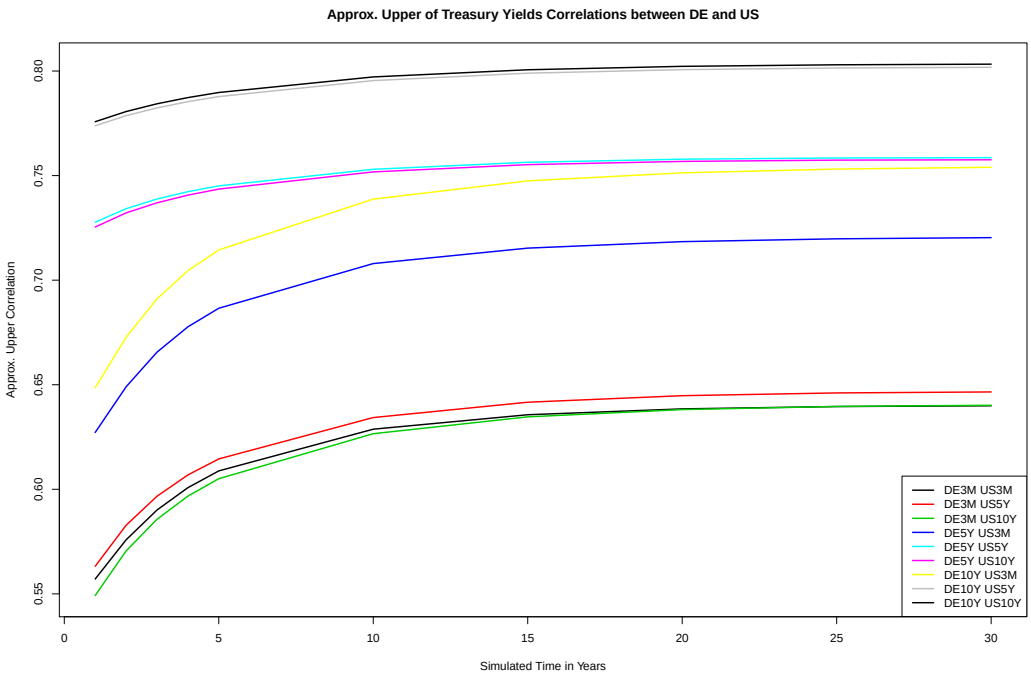


Figure 2: Approx. upper bounds for correlation of the Treasury Yields: DE and US

Table 4: Upper Bounds of Correlation of Treasury Yields: DE and US

	1	2	3	4	5	10	15	20	25	30
DE3M US3M	0.56	0.58	0.59	0.60	0.61	0.63	0.64	0.64	0.64	0.64
DE3M US5Y	0.56	0.58	0.60	0.61	0.61	0.63	0.64	0.64	0.65	0.65
DE3M US10Y	0.55	0.57	0.59	0.60	0.61	0.63	0.63	0.64	0.64	0.64
DE5Y US3M	0.63	0.65	0.67	0.68	0.69	0.71	0.72	0.72	0.72	0.72
DE5Y US5Y	0.73	0.73	0.74	0.74	0.75	0.75	0.76	0.76	0.76	0.76
DE5Y US10Y	0.73	0.73	0.74	0.74	0.74	0.75	0.76	0.76	0.76	0.76
DE10Y US3M	0.65	0.67	0.69	0.70	0.71	0.74	0.75	0.75	0.75	0.75
DE10Y US5Y	0.77	0.78	0.78	0.79	0.79	0.80	0.80	0.80	0.80	0.80
DE10Y US10Y	0.78	0.78	0.78	0.79	0.79	0.80	0.80	0.80	0.80	0.80

Table 5: Lower Bounds of Correlation of Treasury Yields: DE and US

	1	2	3	4	5	10	15	20	25	30
DE3M US3M	0.02	0.02	0.02	0.03	0.03	0.03	0.03	0.03	0.04	0.04
DE3M US5Y	0.02	0.02	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.04
DE3M US10Y	0.02	0.02	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.04
DE5Y US3M	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.04
DE5Y US5Y	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.05
DE5Y US10Y	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.05
DE10Y US3M	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.04
DE10Y US5Y	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.05	0.05	0.05
DE10Y US10Y	0.03	0.03	0.04	0.04	0.04	0.04	0.04	0.05	0.05	0.05

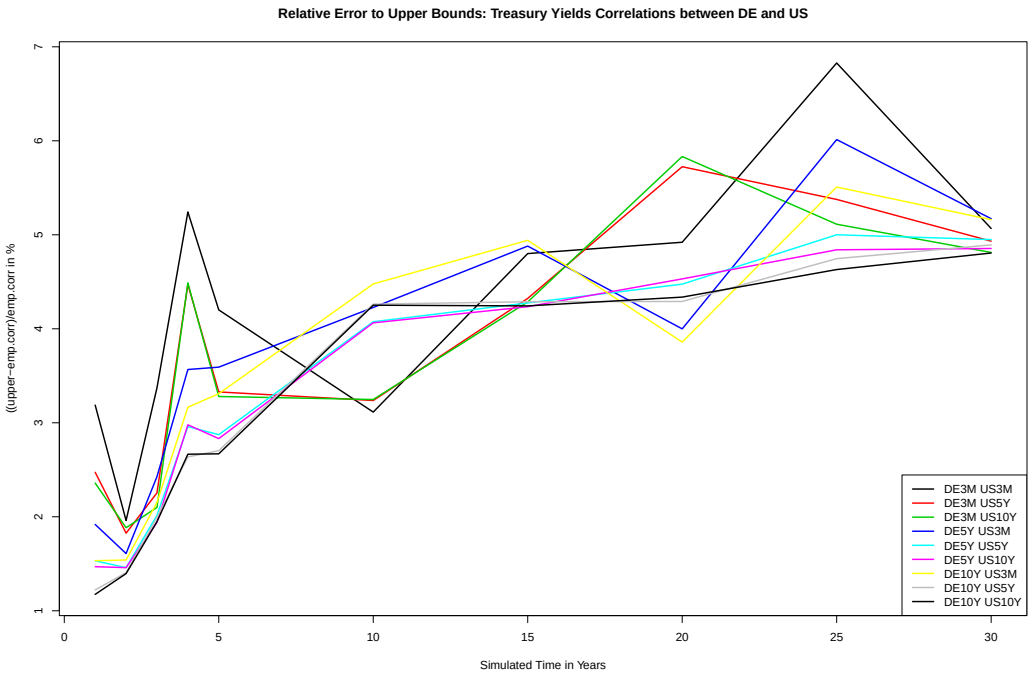


Figure 3: Relative difference between the upper bounds and the empirical correlation of the Treasury Yields: DE and US

3 Validate the dynamic correlation of treasury yields

In the second part of this paper we validate the dynamical (conditional) correlations of the treasury yields across the ESG economies. We will compare the conditional correlations implied by the ESG-Model with those implied by two alternative approaches: the realized correlations and the dynamical-conditional-correlation (DCC).

For the conditional correlations the most important stylized fact among others is the *long memory* property which is normally characterize by the auto-correlation-function (ACF). First we use the heterogeneous autoregressive model (HAR) to fit the realized correlation and use the DCC model (see Engle [2009]) to fit the historical treasury yields of two Economies: Germany and U.S.A. Then we compare the fitted conditional correlations with the conditional correlations of ESG-Model where we use two approximations to extract them from ESG-Model.

3.1 Some empirical facts of historical yields

We use the daily datasets of Germany and U.S.A. treasury yields of maturies 3 months, 5 and 10 years from $1991-10-31$ to $2010-02-23$ from Bloomberg. These three yields are the so-called pivot-yields in ESG-Model and representative enough for our analysis.

Figure 4 shows the historical zero-yields bootstrapped from the par-yields in Bloomberg. Here we use semi-annually coupon payments for both of U.S. and Germany in order to be consistent with ESG-Assumptions although the Germany treasuries (Bundesanleihen) only pay coupons annually.

We are interested in the *conditional correlations* of the treasury yields between Germany and U.S.A.

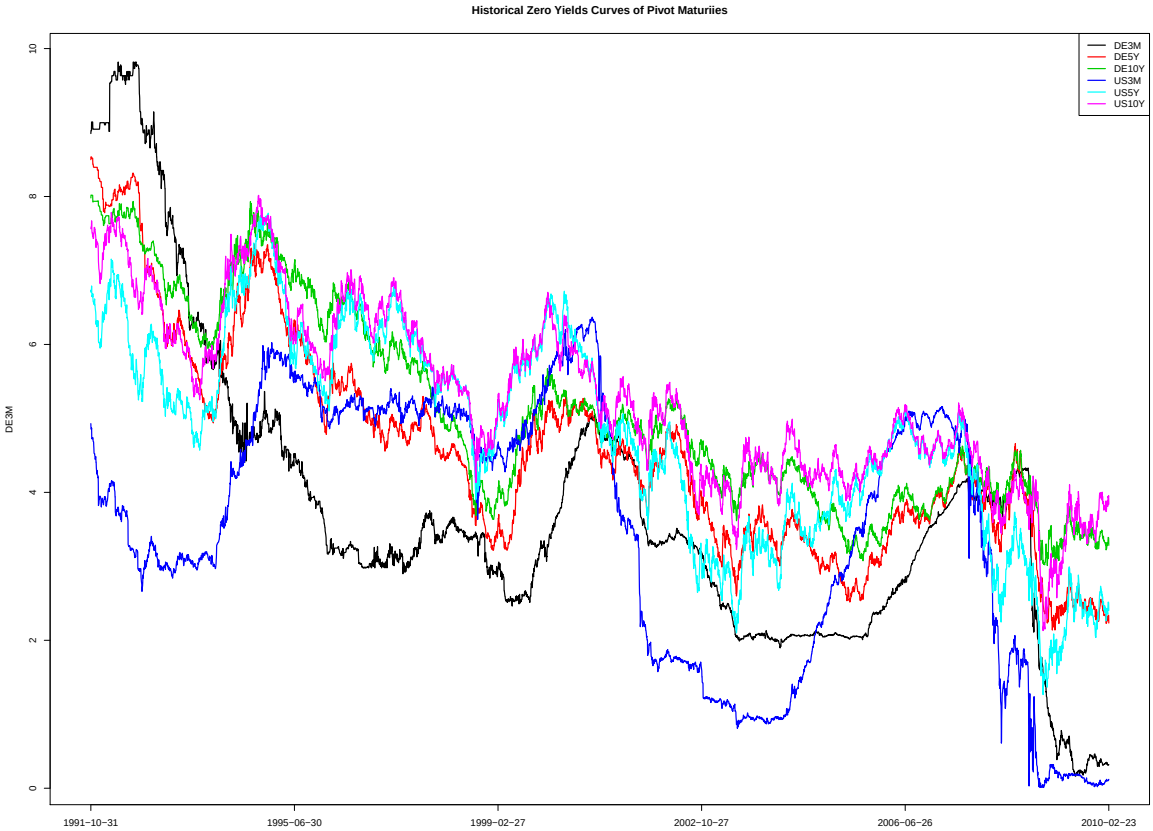


Figure 4: Historical Zero Treasury Yields: DE and US

Let us now define the *conditional covariance* of yields $R^{e1},\ R^{e2}$ of two economies $e1,\ e2$ at time t by

$$\mathbb{C}ov_{1,2}(t;h) := \mathbb{E}_t\left[(R^{e1}(t+h,T) - \mathbb{E}_t[R^{e1}(t+h,T)])(R^{e2}(t+h,S) - \mathbb{E}_t[R^{e2}(t+h,S)])\right], \tag{3.1}$$

where the subscript in $\mathbb{E}_t[.]$ denotes the conditional expectation based on the information up to time t.

Let $Var_1(t;h) := \mathbb{C}ov_{1,1}(t;h)$ be the *conditional variance*. The *conditional correlation* is then defined by (see equation (2.3) of Engle [2009])

$$\rho_{1,2}(t;h) := \frac{\mathbb{C}ov_{1,2}(t;h)}{\sqrt{\mathbb{V}ar_1(t;h)\mathbb{V}ar_2(t;h)}}. \tag{3.2}$$

Similar to the case of modeling and forecasting volatilities, the conditional correlation is not an observable variable. We will analysis the conditional correlation benchmarked on the *realized correlation*, which is a model-free, observable variable of the dynamical evolution of two stochastic processes (see Andersen et al. [2003] and Barndorff-Nielsen and Shephard [2003] for general theoretical backgrounds).

Let $\hat{R}^{e1}(t,T)$ and $\hat{R}^{e2}(t,S)$ be two historical observations of $R^{e1}(t,T)$ and $R^{e2}(t,S)$. The *realized covariance* of $R^{e1},\ R^{e2}$ at time t over a risk horizon h, e.g. one month, is defined as

$$R\mathbb{C}ov_{1,2}(t;h) := \sum_{s=t}^{t+h} \Delta\hat{R}^{e1}(s,T)\Delta\hat{R}^{e2}(s,S), \tag{3.3}$$

where $\Delta\hat{R}_{.}(s,.) := \hat{R}_{.}(s+\delta) - \hat{R}_{.}(s)$ for a fixed small time step δ.

Let $R\mathbb{V}ar_1(t;h) := R\mathbb{C}ov_{1,1}(t;h)$ be the *realized variance*. The *realized correlation* is then defined by

$$R\rho_{1,2}(t;h) := \frac{R\mathbb{C}ov_{1,2}(t;h)}{\sqrt{R\mathbb{V}ar_1(t;h)R\mathbb{V}ar_2(t;h)}}. \tag{3.4}$$

Figure 5a shows the realized correlations of the Germany and U.S. treasury yields for risk horizon of one month. Figure 6a shows the average values of overlapped realized correlations of one month (21 days), one quarter (63 days), half year (126 days) and one year (252 days).

The most important stylized facts of dynamic correlations is the *long-memory* property. We use the *autocorrelation-function (ACF)* to characterize it. Figure 5b shows the ACF of each of the non-overlapped monthly realized correlations. Figure 6b shows the ACF of overlapped realized correlations with risk horizons: one month, one quarter, half and one year. The long-memory property is obviously. Furthermore, in most of the ACF we can observe a regime-switch of the realized correlation in year of 1997 which may reflect the facts of *Federal Reserve Monetary Policy Changes* and introducing of *EURO*.

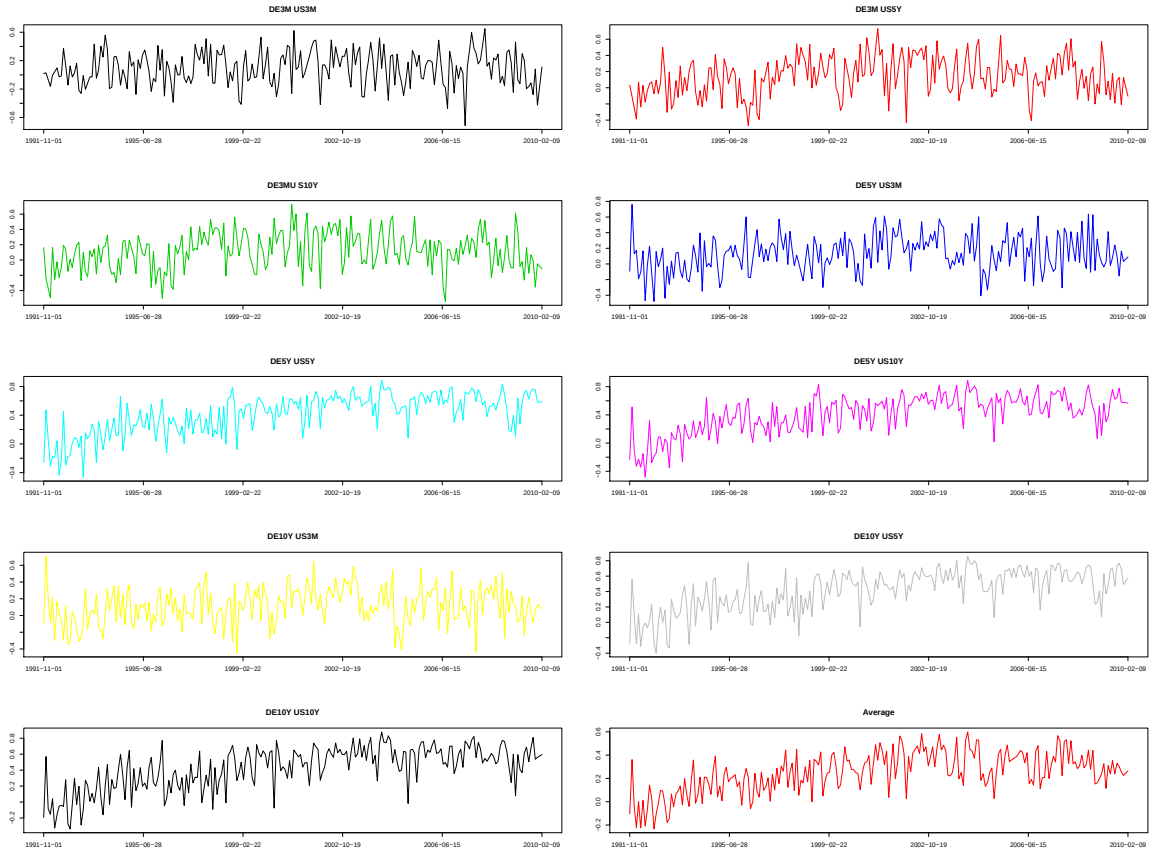
3.2 Some theoretical facts

The fundamental relation between the non-observable conditional correlation and the observable realized correlation can be derived from the following fact

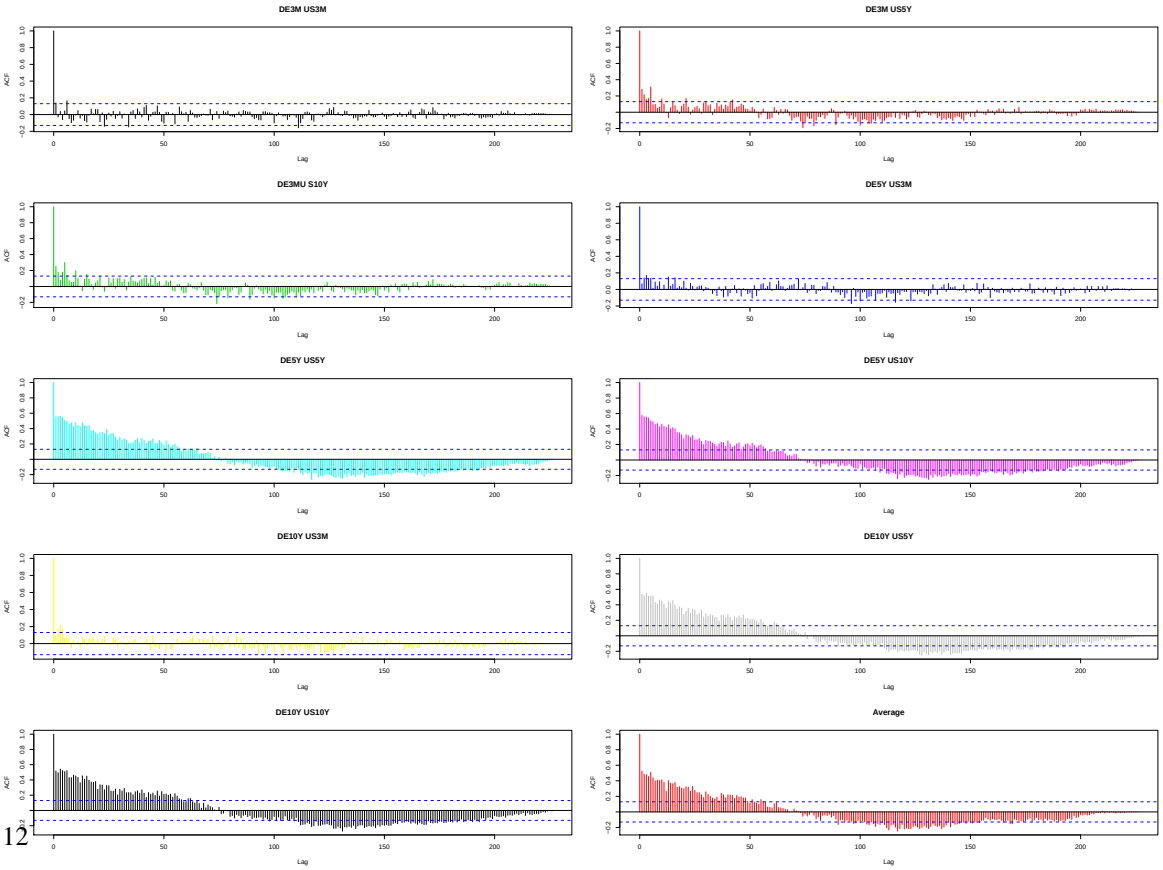
Proposition 3.1. *Let $QCV_{1,2}(t)$ be the quadratic covariation (see definition in $\S II.6$ of Protter [2005] for the definition) of $R^{e1}(t,T)$ and $R^{e2}(t,S)$. Under regular conditions we have that*

- *the realized covariance $R\mathbb{C}ov_{1,2}(0,t)$ converges to the quadratic covariation $QCV_{1,2}(t)$, as the time step δ goes to 0 (see theorem 23 of Protter [2005]).*
- *the conditional covariance $\mathbb{C}ov_{1,2}(t;h)$ equals the conditional expectation of the quadratic covariation for $0 \le t \le t+h$ (see Corollary 1 of Andersen et al. [2003]),*

$$\mathbb{C}ov_{1,2}(t;h) = \mathbb{E}_t[QCV_{1,2}(t+h) - QCV_{1,2}(t)]. \tag{3.5}$$

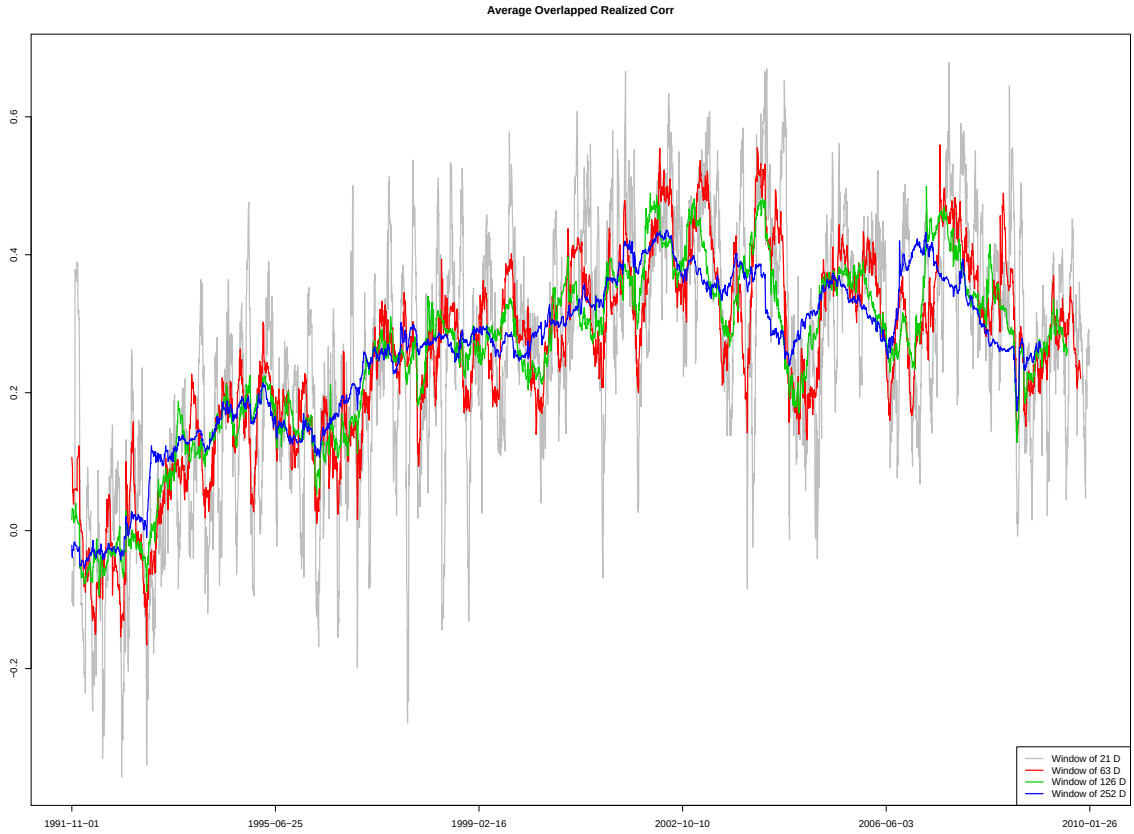


(a) One month realized correlation

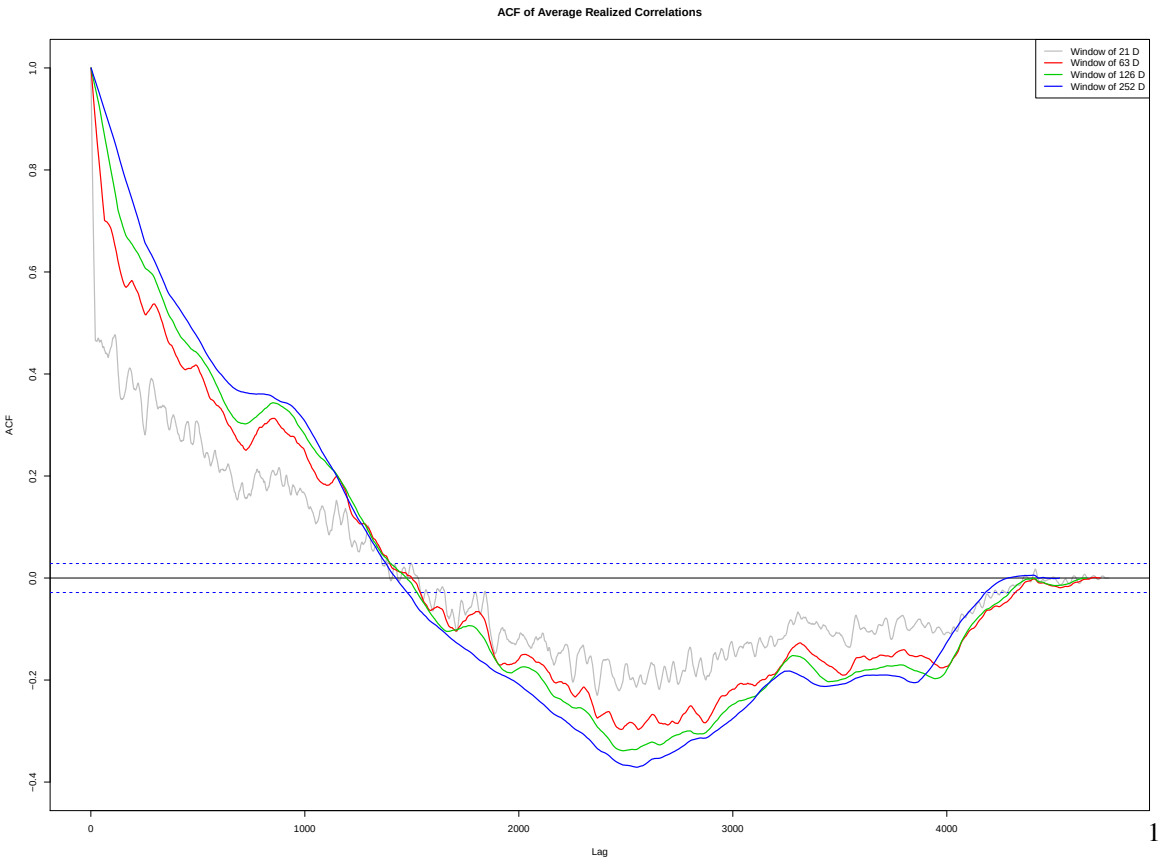


(b) Auto correlation function

Figure 5: One month realized correlation and its ACF



(a) Average overlapped realized correlations



(b) Auto correlation function

Figure 6: Average overlapped realized correlation and its ACF

Remark 3.1. *By a slight abusing of the Proposition 3.1, we can approximate the conditional correlation through the modeling of the realized correlation if we have sufficient small time step (or equivalently sufficient frequent observations) over a risk-horizon, i.e.*

$$\rho_{1,2}(t;h) \approx \mathbb{E}_t \left[R\rho_{1,2}(t;h) \right]. \tag{3.6}$$

3.3 Some models and empirical test results

HAR-Model for Realized Correlation Following Corsi [2006] we use a *heterogeneous autoregressive model* (HAR) to fit the realized correlation. See also Corsi [2004] for general background and Andersen and Benzoni [2008] §4.4.2 for an application on realized volatility of U.S. treasury yields.

We will model the monthly realized correlation as following

$$\rho_{1,2}(t+1;m) = a + b\rho_{1,2}(t;m) + c\rho_{1,2}(t;q) + d\rho_{1,2}(t;s) + \epsilon_{t+1}, \tag{3.7}$$

where $q,\ s$ denote the quarterly and semi-annually (overlapped) realized correlation and ϵ_t the random errors. Similar analysis can be done for quarterly realized correlations by including the annually realized correlation.

The fitted results of this HAR-Model can be found in figure 8a and figure 9a. The ACF of the fitted HAR-Model can be found in figure 8b and 9b. We see that this simple HAR-Model fits the realized correlation with a quite satisfactory quality.

DCC-Model for Conditional Correlations Following Engle [2002] we use the *dynamic conditional correlation GRACH model*(DCC) to model the conditional correlations by fitting the changes of German and U.S. treasury yields. See Engle and Sheppard [2001] for more theoretical details and Engle [2009] for a monographic discussion. Our fitting procedure is base on the *R-Package: ccgarch* (see Nakatani and Teräsvirta [2008] and Nakatani [2009]).

For a $k-$dimensional return process r_t with risk horizon h a typical DCC-Model can be defined as follows:

$$r_{t+h}|\mathcal{F}_t \sim N(0, D_{t+h}R_{t+h}D_{t+h}) \tag{3.8}$$

$$D_{t+h} = diag(\sigma_{i,t+h}) \tag{3.9}$$

$$\sigma^2_{i,t+h} = \omega_i + \kappa_i r^2_{i,t} + \lambda_i \sigma^2_{i,t},\ i = 1, \cdot, k \tag{3.10}$$

$$\epsilon_{t+h} = D^{-1}_{t+h} r_{t+h} \tag{3.11}$$

$$Q_{t+h} = \bar{Q}(1-\alpha-\beta) + \alpha\epsilon_t\epsilon_t^{'} + \beta Q_r \tag{3.12}$$

$$R_{t+h} = diag(Q_{t+h})^{-1/2} Q_{t+h} diag(Q_{t+h})^{-1/2}, \tag{3.13}$$

where $\mathcal{F}_t$ is the filtration up to time t and $\bar{Q}$ is the (unconditional) sample correlation matrix. In this specification, the unconditional mean of Q_t is equal to the sample correlation. In this approach the number of parameters is greatly reduced from $\frac{k^2+5k}{2} + 2$ to $3k+2$ (for the case of k-dimensional GRACH-Model).

Remark 3.2.
- *The conditional correlation $\rho_{1,2}(t;h)$ of yields $R_1(t,T),\ R_2(t,S)$ over risk-horizon h is equal to the conditional correlation of changes of yields $\Delta_1(t,T),\ \Delta_2(t,S)$ because $R_1(t,T),\ R_2(t,S)$ are measurable with respect to filtration $\mathcal{F}_t$.*
- *To get a better fitting result we initialized the parameters by two steps similar to the estimation procedure suggested by Engle and Sheppard [2001]:*
  - *for parameters in conditional variances D_t we fit k single GARCH(1,1) for each (demeaned) yield changes $\Delta_i(t,T) := R_i(t+h,T) - R_i(t,T)$ for two economies $i = 1,\ 2$ and each pivot maturity T.*
  - *for parameters in the conditional correlation Q_t we fit a GARCH(1,1) on the average monthly realized correlations. (An alternative Ansatz could be using the fitted conditional correlation from HAR-Model.)*

The fitted results of this DCC-Model can also be found in figure 8a and figure 9a and the ACF of the fitted DCC-Model can be found in figure 8b and 9b.

Comparing with the realized correlation and HAR-Model we see that the DCC-Model fit the conditional correlations in a plausible manner.

Approximations of ESG-Model Similar to the proposition 2.1 we have the following conditional correlation for treasury yields

Proposition 3.2. ***(Conditional Correlation of yield curves between two economies)***
The conditional correlation over a risk-horizon h for two yield curves $R^{e1}(t,T)$,$R^{e2}(t,S)$ is given by (cf. 3.1)

$$\begin{aligned}
&\mathbb{C}orr_t[R^{e1}(t+h,T+h),R^{e2}(t+h,S+h)]\\
&=\frac{\mathbb{C}ov_t\left[R^{e1}(t+h,T+h),R^{e2}(t+h,S+h)\right]}{\sqrt{\mathbb{V}ar_t\left[R^{e1}(t+h,T+h)\right]\mathbb{V}ar_t\left[R^{e2}(t+h,S+h)\right]}}\\
&=\frac{\mathbb{C}ov_t\left[\sum_{i=1}^3 B_i^{e1}(t,T)X_i^{e1}(t+h),\sum_{i=1}^3 B_i^{e2}(t,S)X_i^{e2}(t+h)\right]}{\sqrt{\mathbb{V}ar_t\left[\sum_{i=1}^3 B_i^{e1}(t,T)X_i^{e1}(t+h)\right]\mathbb{V}ar_t\left[\sum_{i=1}^3 B_i^{e2}(t,S)X_i^{e2}(t+h)\right]}}\\
&=\frac{\sum_{i=1}^3 B_i^{e1}(t,T)B_i^{e2}(t,S)\mathbb{C}ov_t\left[X_i^{e1}(t),X_i^{e2}(t)\right]}{\sqrt{\sum_{i=1}^3(B_i^{e1}(t,T))^2\cdot\mathbb{V}ar_t\left[X_i^{e1}(t)\right]\sum_{i=1}^3(B_i^{e2}(t,S))^2\cdot\mathbb{V}ar_t\left[X_i^{e2}(t)\right]}}\\
&=\frac{\sum_{i=1}^3 B_i^{e1}(t,T)B_i^{e2}(t,S)\big(\mathbb{E}_t[X_i^{e1}(t)X_i^{e2}(t)]-\mathbb{E}_t[X_i^{e1}(t)]\mathbb{E}_t[X_i^{e2}(t)]\big)}{\sqrt{\sum_{i=1}^3(B_i^{e1}(t,T))^2\cdot\mathbb{V}ar_t\left[X_i^{e1}(t)\right]\sum_{i=1}^3(B_i^{e2}(t,S))^2\cdot\mathbb{V}ar_t\left[X_i^{e2}(t)\right]}},
\end{aligned}\tag{3.14}$$

where in the second to last step we use the fact that only the $(W_i^{e1}(t),W_i^{e2}(t))$ are correlated and we use the fact that the Affine-Terms depend only on time-to-maturity.

It is sufficient to calculate the conditional covariance of the related state variables $X_i^{e1},X_i^{e2},\quad i=1,2,3$. The main task now turns out to be calculating the following terms

$$\mathbb{E}_t[X_i^{e1}(t+h)X_i^{e2}(t+h)].$$

Remark 3.3. *To evaluate the conditional correlations we first need to extract the positive initial state variable $X_i^e(t)>0$ for all t by using the historical pivot yield curves and the ESG-Parameters. There are two possible approaches to derive them.*

- *One way is that for a fixed ESG economy and three pivot yields with (time-to-)maturities $T_1,\ T_2,\ T_3$ we solve a 3×3 linear equation system at each time t.*

$$\begin{aligned}
R^e(t,T_1)&=-\frac{\log A^e(t,T_1)-\sum_{i=1}^3 B_i^e(t,T_1)X_i^e(t)}{T_1}\\
R^e(t,T_2)&=-\frac{\log A^e(t,T_2)-\sum_{i=1}^3 B_i^e(t,T_2)X_i^e(t)}{T_2}\\
R^e(t,T_3)&=-\frac{\log A^e(t,T_3)-\sum_{i=1}^3 B_i^e(t,T_3)X_i^e(t)}{T_3}.
\end{aligned}\tag{3.15}$$

- *The other one is to use the updated value of the state variables derived in the Kalman-filtering estimation procedure for the treasury yields model. The results of the two approaches are quite close and here we use the solution from the linear equation above.*

Unfortunately, there are a not-negligible portion of negative state variable in both of two economies. Even after we have tried to get a smaller portion of negative state variables for each single economy, this is still a

serious problem because the time points of negative state variables are disjoint and which yields again a larger proportion of problematic conditional correlations.

Figure 7 shows the historical state variables with recognizing portion of negative values.

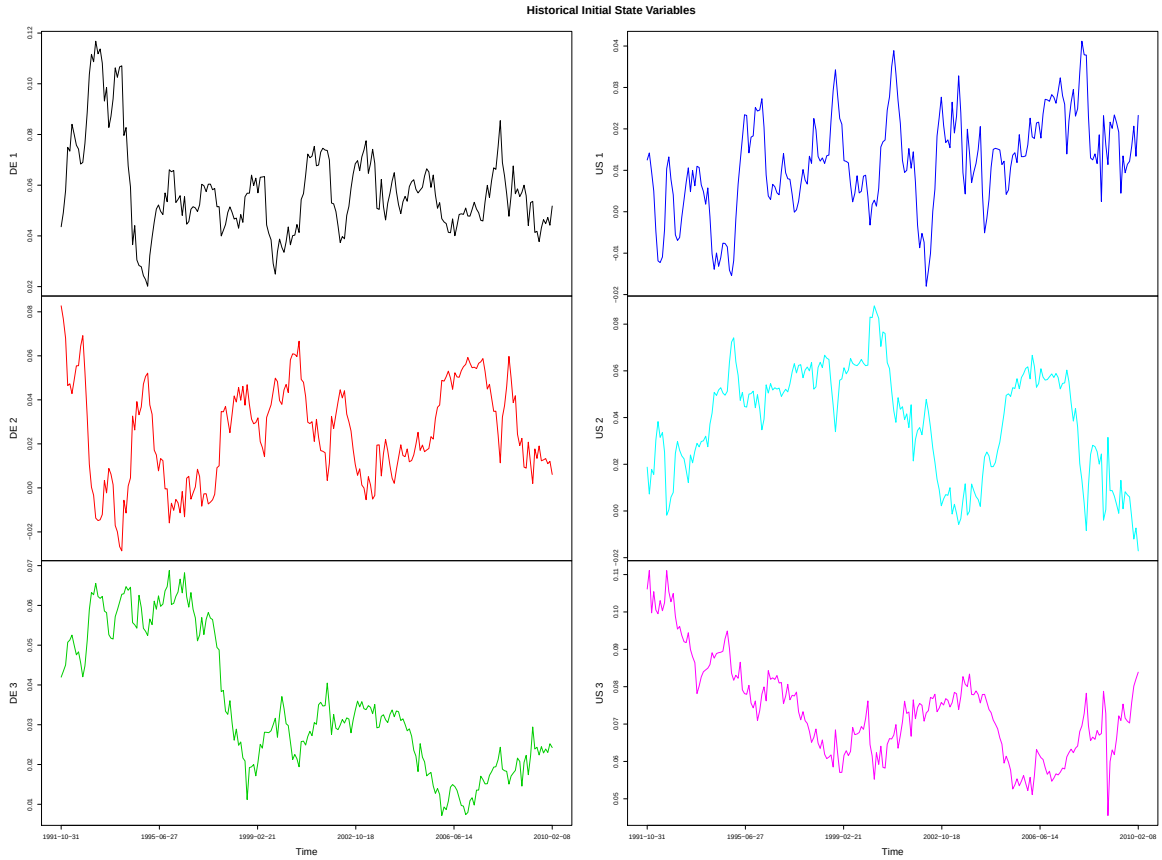


Figure 7: Historical initial state variables based on ESG-Model

So far to our best knowledge, there is no analytical expression about the correlation of ESG yield-curve model, we use two approximations to evaluate the dynamic correlation of treasury yields between ESG economies.

- One is given by an upper-bound of the theoretical value of the correlation based on the method derived in the first part of this paper, see theorem 2.1. This method is quite reliable as shown by the empirical test where the upper-bounds of (unconditional) correlations are only slightly larger than the empirical correlations derived from the ESG-scenarios. We will use this method to detect both of the values and the stylized facts of the conditional correlations of ESG-Model. The positions of problematic conditional correlations caused by the negative initial state variables are computed by (linear) interpolation.

- The other one is derived by assuming a *Gaussian-Mapping* of the ESG-Model. That is, we compute two correlated *three factor Vasicek-Model* (G3++) by using the ESG-Parameters (see section 22.7 of Brigo and Mercurio [2006] for more details about this topic). This **Ansatz** has no theoretical foundation and our empirical test shows that the values of correlations are not so plausible. However, the computing is not affected by the negative initial state variables. We use this method just to detect the stylized facts of the conditional correlations of ESG-Model.

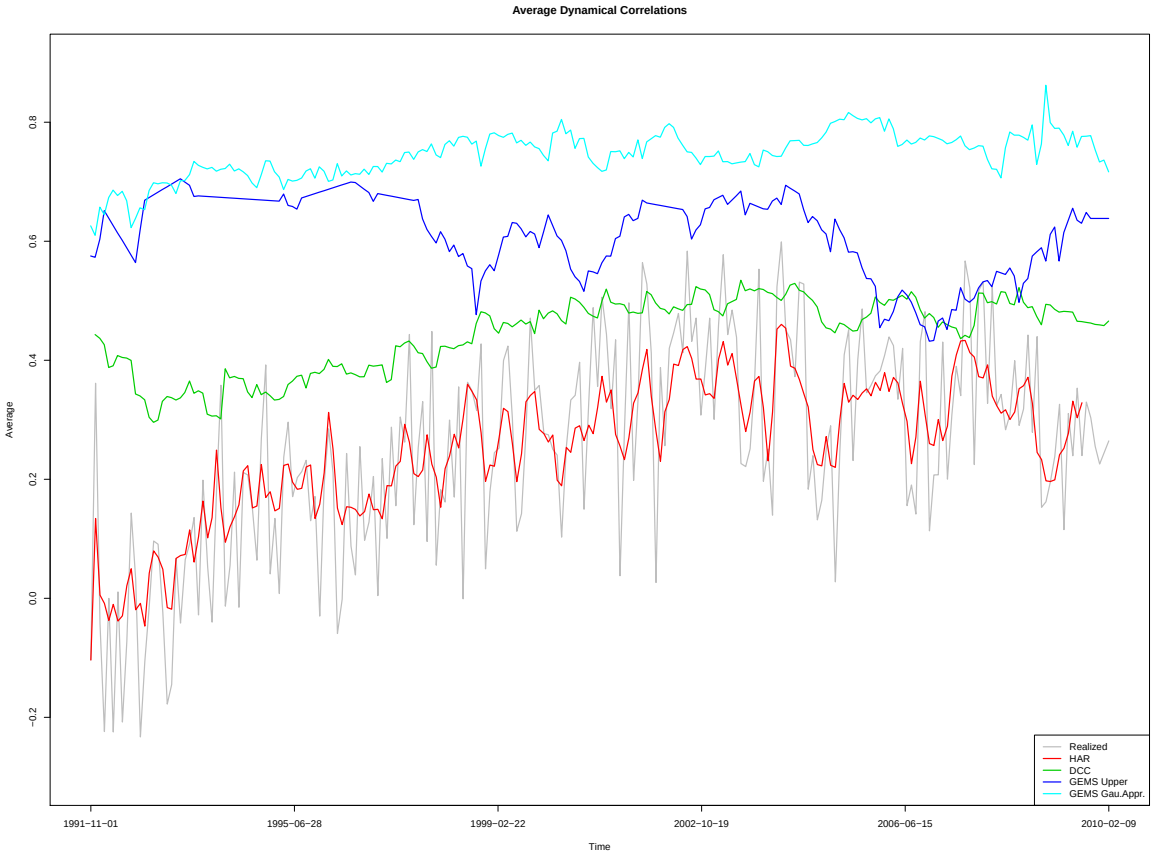
We summarize all of the results in figures 8a and figure 9a and the ACFs in figures 8b and 9b. Following is our finding.

Benchmarked on the results of the (model-free) realized correlations (see HAR-model of (3.7)), we find that the results of DCC-Model (see (3.8)) are in general plausible, both in the values and the long-memory stylized facts. The results of ESG-Model might in general be not so satisfactory. The approximation by Gaussian-Mapping matches the long memory stylized facts quite well but the values of this approximation are too high. The upper bound approximation yields a quite different long memory stylized facts and the values are also quite high although this approximation-method is very reliable.

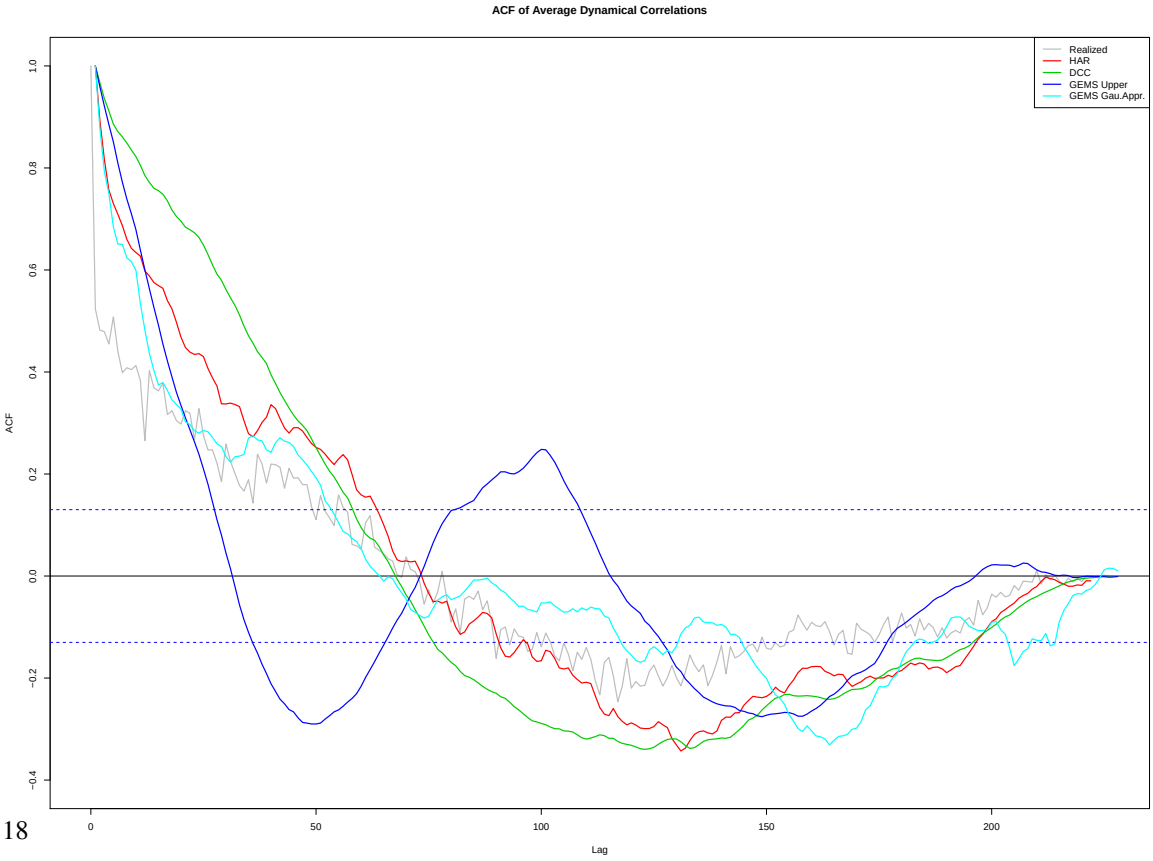
Although comparing with the results of HAR and DCC-Models the conditional correlation based on ESG-Model seems like to be not so satisfiable, we should remark that our requirement on dynamical correlations is much stronger than how the ESG actually models and implements the correlations. The differences of test results are also partially caused by the difficulties in the estimation of ESG-Model to avoid negative state variables for each single economy which can be identified by extracting the initial state variables of ESG-Model using the historical treasury yields (see figure 7).

4 Outlook

- There are various alternative modelling approaches of realized correlations. An very promising approach is the *Square Root Process* (Jacobi-Model). We can compare its fitting results with HAR-Model. However, the implementation is much more involved than fitting HAR-Model. See van Emmerich [2009] and Boortz [2008]. Another one is *Wishart Inverse Covariance Model* (WIC-Model), see Jin and Maheu [2009] where they also compare their results with DCC-Model.
- We can further investigate the fitting results of various models in line with those of Andersen and Benzoni [2008] and Jacobs and Karoui [2008] where they also study the Affine-Models similar to our ESG-model.
- We can also apply similar analysis on the other ESGs such as *Barrie-Hibbert-ESG* and compare their performances with each other.

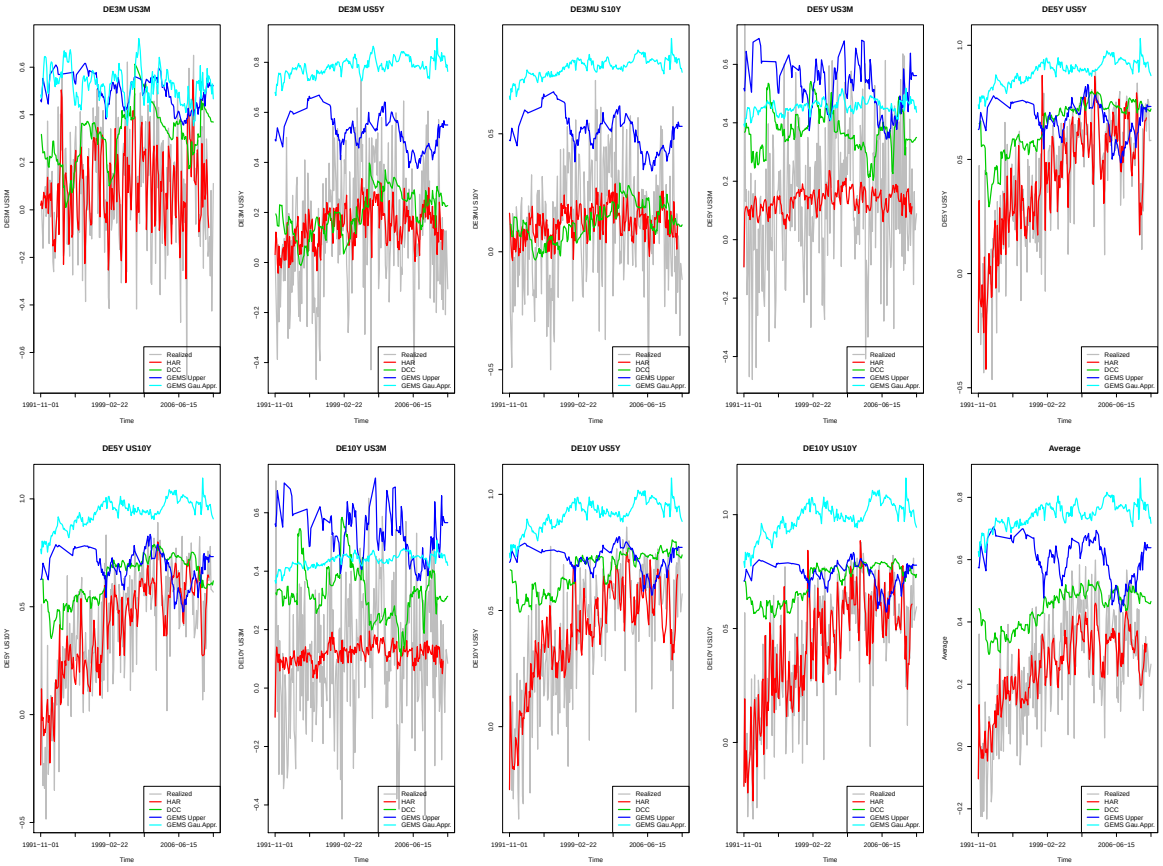


(a) Fitted results of average one-month conditional correlation

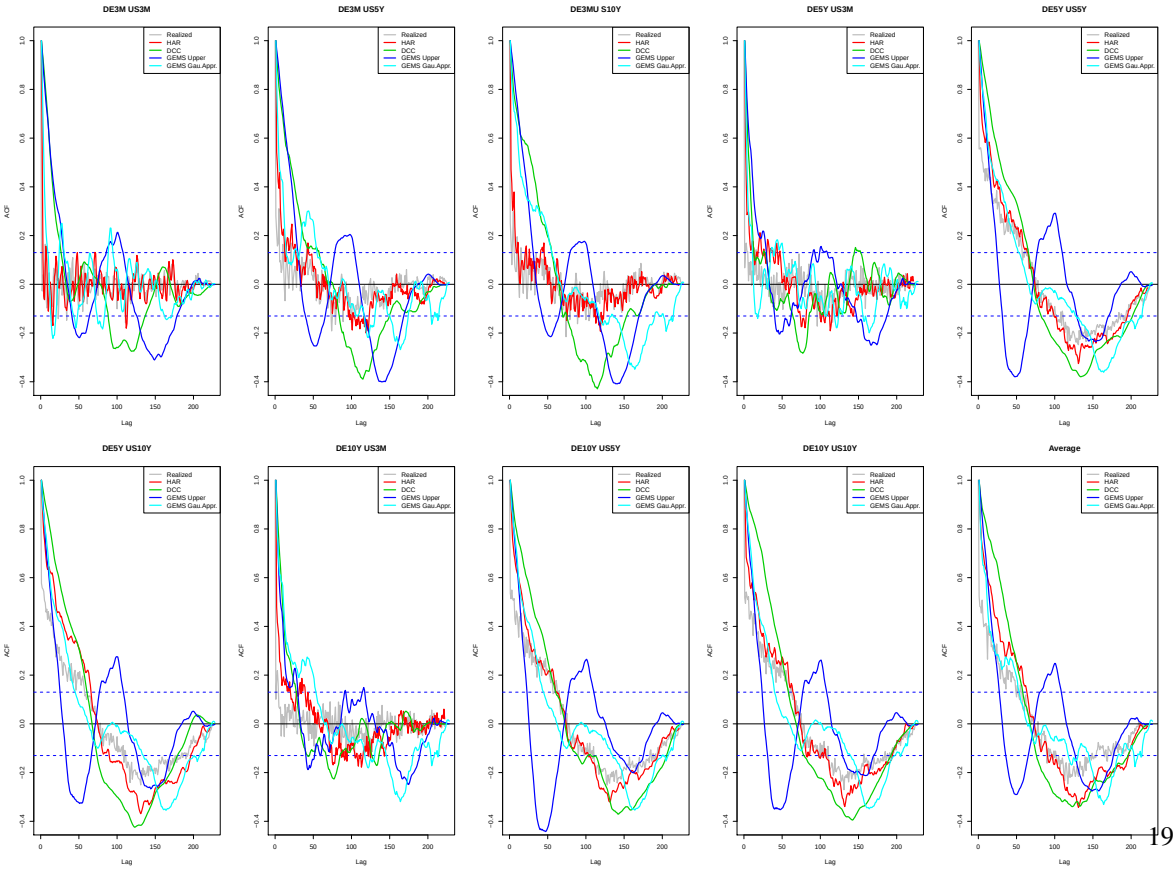


(b) Auto correlation function

Figure 8: Fitted results of average one-month conditional correlation and its ACF



(a) Fitted results of all one-month conditional correlations



(b) Auto correlation function

Figure 9: Fitted results of all one-month conditional correlation and its ACF

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Appendix

Lemma 4.1. *Let $U(t, g)$ be continuous and locally Lipschitz on an open connected set $O \subset \mathbb{R}^2$. Suppose $g(t)$ is a solution to $g'(t) = U(t, g)$ on the interval $[t_0, t_1]$.*

Let $h(t)$ be the solution of the following differential inequality on interval $[t_0, t_1)$

$$h_-'(t) \leq U(t, h(t)),\ h(t_0) \leq g(t_0)$$

where $h_-'(.)$ denotes the right hand derivative of h. Then

$$h(t) \leq g(t) \ \textit{for} \ t \in [t_0, t_1).$$

Proof of this lemma can be found in chapter 6 of Newhouse [2005]. Further details about the *differential inequalities* can be found in section $III.4$ of Hartman [2002].