Optimal Aging with Uncertain Death*

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Abstract. This note extends the theory of optimal aging and death (Dalgaard and Strulik, 2010) towards uncertain death. Specifically, it is assumed that at any age the probability to survive depends on the number of health deficits accumulated. It is shown that the results in Dalgaard and Strulik (2011) on the foundation of the Preston curve (the association between income and life-expectancy across countries) are robust against this extension. While results virtually coincide at high income levels, the stochastic version predicts somewhat more curvature of the Preston curve at low income levels. Taking uncertain death and a precautionary motive for health investment into account thus further improves a bit the anyway good fit of the Preston curve.

Keywords: Aging, Longevity, Health, Savings, Preston Curve.

JEL: D91, J17, J26, I12.

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1. Introduction

Dalgaard and Strulik (2010) have proposed a novel theory of aging and death, which is deeply founded in recent insights from gerontology (Arking, 2006, Gavrilov and Gavrilova, 1991, Mitnitski and Rockwood, 2002). This allowed a reasonable and robust calibration of the model with economic and health data. Using a calibration for a representative US citizen, Dalgaard and Strulik (2010) have shown by way of an out of sample prediction that the theory accounts for about 80% of the Preston curve (the world-wide association between income and life-expectancy) with a causality running from income to health. Simultaneously the theory preserves the insight from Preston (1975) that most of the historical gains in longevity were achieved by shifts of the curve due to medical technological progress.

The model of Dalgaard and Strulik (2010) has assumed that death is a certain event, occuring when sufficiently many health deficits have been accumulated and the human body is too frail to support any longer life. This simplification has allowed them to derive many expressions analytically, a fact that has lead to a rigorous understanding of the biological and behavioral mechanisms at work when the human body ages. But, understandably, it has also opened the door for criticism. In real life the time of death is uncertain and this fact may affect human behavior. The present paper gets rid of this criticism by showing that taking uncertain death into account modifies the original results only marginally. Following the literature we assume a market for annuities such that precautionary saving for unexpected longevity is not an issue (Sheshinski, 2008). But uncertain death introduces a motive for precautionary investment in health when individuals anticipate that that these investments reduce the probability to die at a certain time. The present paper shows that this mechanism has little power in modifying the Preston curve at low income levels and virtually none at high income levels.

In order to be brief I refer to Dalgaard and Strulik (2010) for an introduction to the problem of health investment and biological aging as well as a discussion of the related literature.

2. The Model

2.1. **The optimization problem.** Consider an adult maximizing utility from consumption $c(t)$ over his or her life. The initial age is for convenience normalized to zero. Let $\rho \geq 0$ denote the rate of pure time preference and $S(t)$ the probability to survive beyond age t. Facing uncertain death, rational individuals calculate the expected utility from life-time consumption by multiplying the instantaneous utility experienced at age t with the probability to survive beyond age t (Kamien and Schwartz, 1991, Ch. 9). The present value of expected utility experienced over the life cycle is thus given by

$$\int_0^T S(t)\mathrm{e}^{-\rho t}u(c(t))\ \mathrm{d}t, \tag{1}$$

with instantaneous utility $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ for $\sigma \neq 1$ and $u(c) = \log(c)$ for $\sigma = 1$.

As in Dalgaard and Strulik (2010) we assume that the individual receives a fixed income stream w through life (see Strulik, 2011, for an extension towards education, retirement, and life cycle wages). Income can be spend on consumption goods c or on health goods h. The relative price of health goods is p. Besides spending income on final goods, the individual may invest in capital k and receive a net interest rate r. The individual takes all prices as given and – in order to keep the problem tractable – there exists no other uncertainty except the time of death. The law of motion for individual wealth is thus given by (2).

$$\dot{k}(t) = w + (r+m)k(t) - c(t) - ph(t). \tag{2}$$

Here we have assumed perfect annuities such that the interest rate is the sum of the rate of return on capital plus the instantaneous mortality rate $m = -\dot{S}/S$. Given the annuity market, individuals inherit no wealth $k(0) = 0$ and leave no bequest $k(T) = 0$. Capital left over at death is distributed among the survivors by the annuity supplier. We thus implicitly assume that the individual is surrounded by sufficiently many other individuals of the same age.

As in the basic model health deficits evolve according to (3).

$$\dot{D}(t) = \mu\left(D(t) - a - Ah(t)^{\gamma}\right), \tag{3}$$

in which μ denotes the force of aging (the rate of "natural" bodily decay), a controls for environmental influences, A specifies general efficiency of the medical technology, $h(t)$ is health expenditure at age t, and $\gamma < 1$ is the health elasticity of health spending.

As an extension to the basic model we now allow death to hit the individual randomly, depending on the health status. Specifically, we assume that the probability to survive beyond age t, $S(t)$, is a simple negative function of the number of accumulated deficits (the frailty index number) at age t:

$$S(t) = \phi\left[1 - \mathrm{e}^{\frac{D-\bar{D}}{\alpha}}\right]. \tag{4}$$

The parameter α controls how rectangular the survival curve is. With increasing α survival becomes more and more rectangular and for $\alpha \to 0$ the model converges to the model of Dalgaard and Strulik (2010), in which all individuals die at the moment when $\bar{D}$ health deficits have been accumulated. Here, most individuals die earlier, with less then the upper limit of possibly health deficits. This means, intuitively, that the severity of deficits is stochastic. Most individuals begin with accumulating mild health deficits (poor vision) first before they acquire severe deficits (cancer) at a later age. But some individuals get the severe deficits first and die prematurely. In this sense the endogenous upper bound T is the maximum life-span that the individual wants to achieve. The parameter ϕ controls the probability to die without any health deficit (for example by traffic accident).

The problem is to maximize (1) subject to (2)–(4), the initial conditions $D(0) = D_0$, $k(0) = 0$, and the terminal conditions $D(T) = \bar{D}$ and $k(T) = 0$. That is, at the maximum life-span individuals spend the last dollar before they expire without leaving any wealth to the life insurer (supplier of annuities). The problem can be solved by employing optimal control theory; the state variables are $k(t)$ and $D(t)$ and the control variables are consumption $c(t)$ and health investments $h(t)$. The individual takes the impact of health deficits on survival into account but, in line with the literature, we assume that the annuity is a price uncontrollable by the individual like all other prices. This assumption is not essential but simplifies the problem and provides a solution for the consumption path that is in line with the conventional literature on stochastic survival. Following Strulik (2011) the model can be easily extended to take health deficits in utility into account but for the ease of exposition we treat health deficits purely functional. Individuals don't like them because they reduce the probability to survive and thus longevity.

2.2. **Optimal Aging.** The associated Hamiltonian is

$$H = S\mathrm{e}^{-\rho t}\frac{c^{1-\sigma}-1}{1-\sigma} + \lambda_k\left[w + (r+m)k - c - ph\right] + \lambda_D \mu \left[D - aAh^{\gamma}\right].$$

The first order conditions are

$$\frac{\partial H}{\partial c} = S\mathrm{e}^{-\rho t}c^{-\sigma} - \lambda_k = 0 \tag{5}$$

$$\frac{\partial H}{\partial h} = -\lambda_k p - \lambda_D \mu \gamma A h^{\gamma-1} = 0 \tag{6}$$

$$\frac{\partial H}{\partial k} = \lambda_k(r+m) = -\dot{\lambda}_k \tag{7}$$

$$\frac{\partial H}{\partial D} = \frac{\partial S}{\partial D}\mathrm{e}^{-\rho t}\frac{c^{1-\sigma}-1}{1-\sigma} + \lambda_D\mu = -\dot{\lambda}_D. \tag{8}$$

From log-differentiating (5) with respect to age and (7) we obtain the Euler equation for consumption:

$$g_c \equiv \frac{\dot{c}}{c} = \frac{r-\rho}{\sigma}. \tag{9}$$

As in the conventional literature, the presence of perfect annuities implies that the Euler equation with uncertain death is the same as under certainty. The reason is that $m = -\dot{S}/S$, implying that the interest rate and the effective discount factor are risen by the same rate ($r+m$ and $\rho+m$) and thus the death rate cancels out.

From log-differentiating (6) wrt age we obtain

$$\frac{\dot{\lambda}_k}{\lambda_k} = \frac{\dot{\lambda}_D}{\lambda_D} + (\gamma-1)\frac{\dot{h}}{h}. \tag{10}$$

And from inserting (5) and (6) into (8) we get

$$-\frac{\dot{\lambda}_D}{\lambda_D} = \mu + \mu\frac{A\gamma h^{\gamma-1}}{p}\cdot\frac{c-c^{\sigma}}{1-\sigma}\cdot\frac{\partial S/\partial D}{S}. \tag{11}$$

Inserting (7) and (11) into (10) we arrive at the "Health Euler" equation:

$$g_h \equiv \frac{\dot{h}}{h} = \frac{1}{1-\gamma}\left\{ r+m-\mu+\mu^2\frac{A}{p}\gamma h^{\gamma-1}\frac{c-c^{\sigma}}{1-\sigma}\frac{\mathrm{e}^{\frac{D-\bar{D}}{\alpha}}}{\alpha\left[1-\mathrm{e}^{\frac{D-\bar{D}}{\alpha}}\right]}.\right\} \tag{12}$$

Compared to Dalgaard and Strulik (2010) the Health Euler is modified by two terms, the annuity term m and a correction term of second order (μ^2). For certain survival up to $\bar{D}$ health deficits we have $m = 0$ and $\partial S/\partial D = 0$ and the model collpases to the basic model. This means that the difference between the stochastic and the basic model vanishes as the survival function becomes more rectangular.

It is also interesting to observe that both terms originating from stochastic death increase g_h. For any given initial conditions there is more precautionary investment in health.

At terminal time T we have $D(T) = \bar{D}$ and the associated Hamiltonian assumes the value of zero indicating that – taken the costs into account – it is not worth to live any longer. Inserting the costate variables from (5) and (6) into the Hamiltonian and noting that $S(\bar{D}) = 0$ shows that the condition that $H = 0$ is automatically fulfilled in the terminal state.

3. Calibration

Identification of the optimal life cycle trajectories is somewhat harder than for the basic model. The reason is the complicated expression for optimal health expenditure (12), which prevents that the time paths for $h(t)$ and $D(t)$ can be obtained analytically as it was the case in Dalgaard and Strulik (2010).

We thus approximate the solution in the following way. We begin with obtaining the optimal T under certainty. This value serves as a first estimate of optimal life-span. We then start at the terminal state and integrate the four dimensional system (2), (3), (9), and (12) backwards using the method of Brunner and Strulik (2000). Since the trajectories representing the optimal solution of the stochastic problem originated from the endpoint of the deterministic solution they do not arrive after T years at the given initial conditions $D(0)$ and $k(0)$ in the first try. We thus iteratively adjust the estimate of terminal point $c(T), h(T)$ until they do.

Figure 1 shows the solution for a "benchmark run" for which we kept as many parameters from the deterministic model as possible. In particular we kept the values for a, A, ρ and μ. Naturally, the value of $\bar{D}$ has to be adjusted because it no longer signifies the number of health deficits with which all individuals die but the health deficits which the longest living individuals in the population accumulate. In other words, the former $\bar{D}$ value of 0.1 from the calibration of the deterministic model is now the frailty index at which 20 year old individuals *expect* to die. The new upper bound $\bar{D}$ is higher.

We estimate $\bar{D}$ together with the parameters α and ϕ from the survivorship function $S(t)$ such that the implied survival rates approximate the real distribution of survival in the US. The notion hereby is that what appears from the individual viewpoint as the probability to actually survive up to age t is, from the aggregate viewpoint, the share of male 20 year old individuals who survive to age t. We calculate this value from the Gompertz Makeham estimate for US males in Strulik and Vollmer (2010). This provides the estimates $\bar{D} = 0.14$, $\alpha = 0.05$, and $\phi = 1.065$. The resulting trajectory and the data (stars) is shown in the lower left panel of Figure 1. We see that the US survivorship function is still quite far away from being rectangular and that the calibrated model approximates the actual survival rate by age in the US quite well.

For an interpretation it is important to note that the model does not deliver survivorship as a function of age (as other economic models of probabilistic survival, see e.g. Heijdra and Romp, 2008). Instead survival is an endogenous function of the health state of the individual, i.e. the number of the accumulated health deficits. This notion is in line with the modern biological theory of aging (see e.g. Arking, 2006, Gavrilov and Gavrilova, 1992): while the process of dying is inherently stochastic, the expected time of death of an individual is not determined by his *age* but by his *health status*, that is by the number of accumulated health deficits.

In order to approximate the real evolution of health deficits (according to Mitnitski and Rockwood, 2002) two more parameters of Dalgaard and Strulik (2010) have to be (mildly) adjusted. The interest rate r is reduced from 6 to 5.95 percent and the curvature parameter σ is raised from 1 (log utility) to 1.1. While the predicted trajectory for health deficits (upper left panel) approximates the data quite well, the predicted trajectory for the share of health expenditure ϵ_h (lower right panel) could still be improved. Compared to the basic model, however, the health expenditure share at high ages is better approximated. The reason is that, given the annuity market, the (relatively few) individuals who actually reach a high age have the funds to spend a lot on health.[1]

An interesting observation is the flattening wealth curve at high ages. The reason is that most people are dead at age 90 and above. This allows the annuity supplier to pay a very high annuity premium to the survivors. The high interest rates in turn mean that long-living individualy can finance a lot of health operations with relatively little wealth.

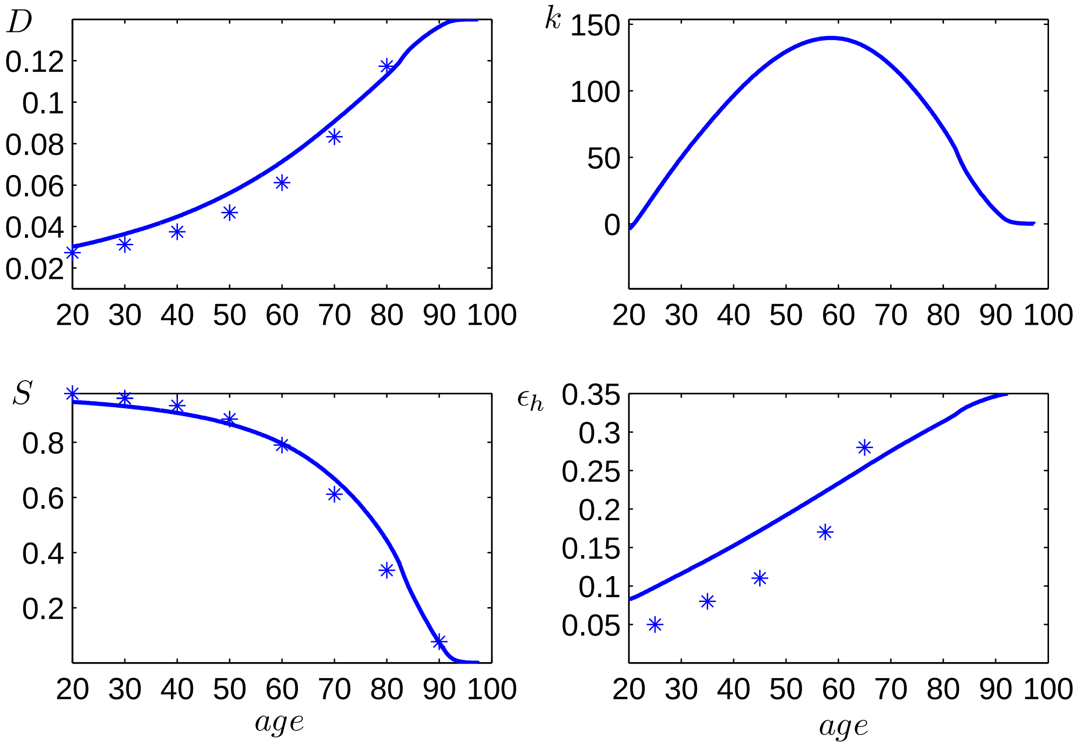
As in Dalgaard and Strulik (2010) the average male US citizen at age 20 expects to expire at age 75 with a frailty index of 0.1. Actually, however, many people pass away earlier and many live substantially longer. Thanks to the early expiry of their contemporaries the long-living ones have the funds to finance health repair, a fact, which allows them to live even longer. Comparing costs and benefits, however, almost everybody prefers to die before the frailty index reaches $\bar{D} = 0.14$, which is, according to the calibration at an age of 102 years.

4. The Preston Curve

We next investigate how uncertain death modifies the main result of Dalgaard and Strulik, the theoretical foundation of the Preston curve. For that purpose, the dashed green line in Figure

[1]The frailty data is taken from Mitnitski and Rockwood, 2002 and the data on health spending by age is inferred from Mazzocco, M. and Személy (2010), see Dalgaard and Strulik (2010) for details.

FIGURE 1: OPTIMAL AGING AND DEATH: STOCHASTIC VERSION

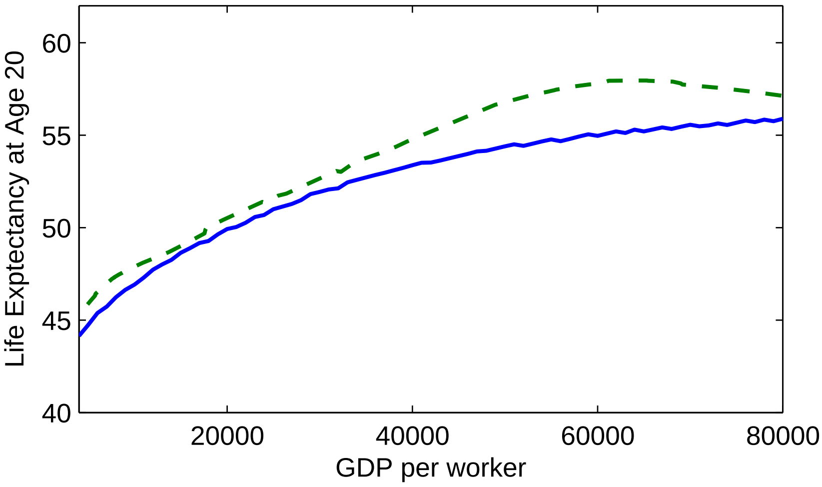


Parameters: $a = 0.013$, $A = 0.0015$, $\mu = 0.043$, $\rho = 0.06$, $r = 0.0595$, $\sigma = 1.1$, $D_0 = 0.0274$, $\bar{D} = 0.14$, $p = 1$, $\alpha = 0.05$, $\phi = 1.065$. stars: data, see main text and Dalgaard and Strulik (2010) for details.

2 re-iterates the Preston curve for life-expectancy at age 20 estimated by Dalgaard and Strulik (2010) for 65 countries. Next we feed the income levels (PPP GDP per worker in 2000) for the 65 countries through the model, keeping the parameters we calibrated to US data fixed, and compute – by numerical integration of the associated path of $S(t)$ – the implied life-expectancy at age 20. The result is shown by the solid blue line in Figure 2. At high and intermediate income levels there is basically no difference discernable between the prediction from the stochastic and the deterministic model. To better corroborate this claim Figure 3 reiterates the original, deterministic estimate.

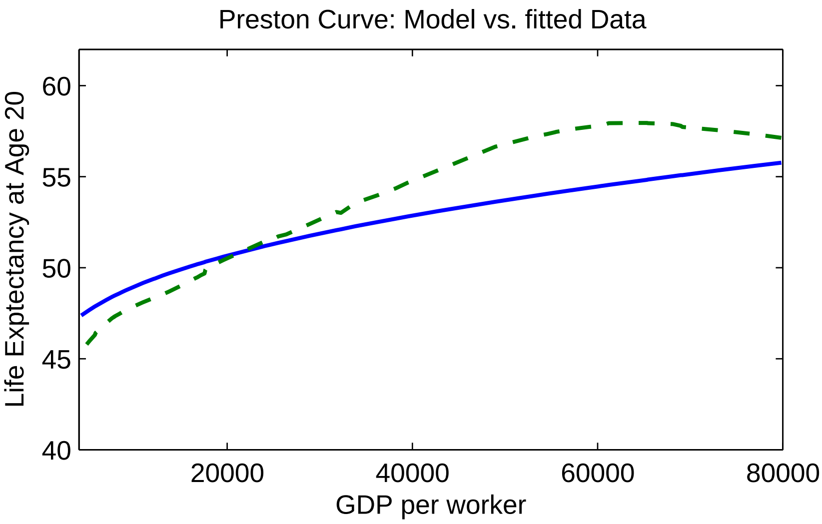
At low income levels the stochastic version predicts more curvature of the Preston curve and approximates the data a bit better. Below an annual income level of \$ 20000 the originally predicted Preston curve is "too flat" while the stochastic version is somewhat "too steep", indicating that (the absence of) precautionary health investments plays a role at low income levels. At low income levels (when marginal utility experienced from "period"consumption is high) individuals are more willing to take the risk of an early death and invest too little in

Figure 2: Preston Curve: Estimate vs. Prediction from Stochastic Model



The figure compares the empirically estimated Preston curve (dotted) for the year 2000 to the Preston curve predicted by the model of optimal aging and uncertain death (solid).

Figure 3: Preston Curve: Estimate vs. Prediction from Deterministic Model



The figure is taken from Dalgaard and Strulik (2010).

health in favor of more current consumption. At high income levels (when marginal utility from period consumption is low) and most further gains in life-time utility originate from experiencing consumption for "another period", the precautionary motive seems to play no role anymore.

5. Conclusion

Since the time of death is uncertain, a naturally arising question is whether a theory of aging that abstracts from this fact approximates reality sufficiently well. In case of the Dalgaard-Strulik (2010) model of optimal aging and death this question can be answered in the affirmative. In particular when income is high, there is virtually no difference between human behavior predicted by the simpler, deterministic model and the more complex, stochastic version.

There is a reason why most dynamic economic models (and most health economic models) abstain from explicitly considering uncertainty. Taking uncertainty into account has a price, which has also been paid by the present study: some previously simple and elegant expressions get bulky and complicated, some expressions can no longer be derived analytically, and the solution trajectories for optimal life-time behavior have to be obtained by numerical approximation instead of by explicit calculus. Explicit analytical solutions however are inevitable for a rigorous understanding of the underlying mechanism in any model. This trade-off and the fact of little quantitative difference of results, suggests the conclusion that the simple, deterministic model is the preferable one.

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