On the Memory of Products of Long Range Dependent Time Series*

Christian Leschinski

Leibniz University Hannover

February 3, 2016

Abstract

This paper derives the memory of the product series $x_t y_t$, where x_t and y_t are stationary long memory time series of orders d_x and d_y, respectively. Special attention is paid to the case of squared series and products of series driven by a common stochastic factor. It is found that the memory of products of series with non-zero means is determined by the maximal memory of the factor series, whereas the memory is reduced if the series are mean zero.

Key words: Long Memory · Products of Time Series · Squared Time Series · Fractional Cointegration

JEL classification: C22; C10

1 Introduction

Products of time series occur frequently in non-linear models such as the bilinear model, random coefficient models, or multiplicative noise models and they also play an important role as interaction terms in time series regressions. Therefore, it is of interest how time series properties such as long range dependence are translated from the factor series x_t and y_t to the product series $z_t = x_t y_t$. In this paper, it is shown that the transmission of memory critically depends on the means of the processes. While the memory of products is the maximum of the memory orders of the factor series if the means are non-zero, the memory order in the product series will be reduced for zero mean processes.

In a related literature Granger and Hallman (1991) and Corradi (1995) have studied the properties of non-linear transformations of integrated variables. For long memory time series Dittmann

*I would like to thank Philip Bertram, Robinson Kruse, Philipp Sibbertsen and Michael Will, for their comments on earlier versions of this paper.

and Granger (2002) have derived the memory properties for transformations of zero mean time series if the transformation can be expressed as a finite sum of Hermite polynomials. The memory of products of long memory time series, however, has not been covered.[1]

The expectation of the product series $z_t = x_t y_t$ that we are interested in, is given by $E[z_t] = \sigma_x \sigma_y \rho_{xy} + \mu_x \mu_y$, where μ_x and μ_y denote the means of x_t and y_t, respectively, σ_x and σ_y are the standard deviations, and ρ_{xy} denotes the correlation between the two series. For general random variables x and y, with finite first and second moments, Goodman (1960) derived the variance of xy. Later, Bohrnstedt and Goldberger (1969) derived the exact covariance of the products xy and vw, where v and w form a second pair of random variables that fulfills the same moment conditions as x and y. According to Bohrnstedt and Goldberger (1969), the variance of z_t is given by

$$\begin{aligned}\sigma_z^2 = \mu_x^2 \sigma_y^2 + \mu_y^2 \sigma_x^2 + E[(x-\mu_x)^2(y-\mu_y)^2] \\ + 2\mu_x E[(x-\mu_x)(y-\mu_y)^2] \\ + 2\mu_y E[(x-\mu_x)^2(y-\mu_y)] \\ + 2\mu_x \mu_y \sigma_x \sigma_y \rho_{xy} - \sigma_x^2 \sigma_y^2 \rho_{xy}^2.\end{aligned}$$

The autocovariance function of the product of two weakly stationary time series x_t and y_t was derived by Wecker (1978). If both series are Gaussian, it is given by

$$\begin{aligned}\gamma_{xy}(\tau) =& \mu_x^2 \gamma_y(\tau) + \mu_y^2 \gamma_x(\tau) + \mu_x \mu_y [\xi(\tau) + \xi(-\tau)] \\ &+ \gamma_x(\tau)\gamma_y(\tau) + \xi(\tau)\xi(-\tau),\end{aligned} \tag{1}$$

where $\xi(\tau)$ denotes the cross-covariance function at lag τ defined as $\xi(\tau) = E[(x_t - \mu_x)(y_{t-\tau} - \mu_y)]$. In the remainder of this paper, the memory properties of the product series z_t will be derived from the asymptotic behavior of (1), as $\tau \to \infty$. Definitions, assumptions and the main result are given in Section 2. Sections 3 and 4 extend these results to squares of long memory series and products of variables with common long range dependent factors. Conclusions are drawn in Section 5.

[1] Note that the application of a log-transformation does not mitigate this issue. It merely converts the problem into determining the memory of the sum of non-linearly transformed series, but the logarithm cannot be represented as required by Dittmann and Granger (2002). Therefore, the memory of the log-transformed series is unknown - as is that of the product.

2 Persistence of Products of Long Memory Time Series

In the following, a time series x_t is a long memory series with parameter d_x if its spectral density $f_x(\lambda)$ at frequency λ obeys the power law

$$f_x(\lambda) \sim g_x(\lambda)\lambda^{-2d_x}, \tag{2}$$

as $\lambda \to 0_+$, or if its autocovariance function $\gamma_x(\tau)$ at lag τ is

$$\gamma_x(\tau) \sim G_x(\lambda)\tau^{2d_x-1}, \tag{3}$$

for $\tau \to \infty$. Here, $g_x(\lambda)$ and $G_x(\lambda)$ denote functions that are slowly varying at zero and infinitiy, respectively. As Beran et al. (2013) show, these definitions are equivalent under fairly general conditions. Hereafter, we write $x_t \sim LM(d_x)$ if x_t fulfills at least one of (2) or (3). For simplicity, we will treat g_x and G_x as constants. The properties of any x_t that is $LM(d_x)$ depend on the value of $d_x \in (-1/2, 1/2)$. For $d_x < 0$, the process is antipersistent, and $f_x(0) = 0$. If $d_x = 0$, $f_x(0) = g_x$ and the process has short memory. Finally, for $d_x > 0$, x_t is long range dependent.

Here, we follow Dittmann and Granger (2002) and distinguish between fractional integration and long memory. The reason is, that we derive the memory of $z_t = x_t y_t$ based on the behavior of $\gamma_{xy}(\tau)$ for large τ that is of the form specified in (3), so that its spectral density is of the form given in (2). A fractionally integrated process $\tilde{z}_t$, on the other hand, has spectral density $f_{\tilde{z}}(\lambda) = |1 - e^{i\lambda}|^{-2d_{\tilde{z}}} g_{\tilde{z}}(\lambda)$, so that $f_{\tilde{z}}(\lambda) \sim g_{\tilde{z}}|\lambda|^{-2d_{\tilde{z}}}$, as $\lambda \to 0_+$, since $|1 - e^{i\lambda}| \to \lambda$, as $\lambda \to 0_+$. While fractional integration is therefore a special case of long memory, the results given here only allow to draw conclusions about the memory properties of the product series.

For the main result we require the following assumptions.

Assumption 1. *$x_t \sim LM(d_x)$ and $y_t \sim LM(d_y)$ are weakly stationary and causal Gaussian processes, with $0 \leq d_x, d_y < 0.5$ and finite second order moments.*

Assumption 2. *If $x_t, y_t \sim LM(d)$, then $x_t - \psi_0 - \psi_1 y_t \sim LM(d)$ for all $\psi_0, \psi_1 \in \mathbb{R}$.*

Assumption 1 is a simple regularity condition, whereas Assumption 2 precludes the presence of fractional cointegration. This will be relaxed in Section 4, where the case of a common long memory factor driving x_t and y_t is considered.

Given these assumptions, the memory of the product series $x_t y_t$ is characterized by the following proposition.

Proposition 1. *Under Assumptions 1 and 2 the product series $z_t = x_t y_t$ is $LM(d_z)$, with*

$$d_z = \begin{cases} \max(d_x, d_y), & \text{for } \mu_x, \mu_y \neq 0 \\ d_x, & \text{for } \mu_x = 0, \mu_y \neq 0 \\ d_y, & \text{for } \mu_y = 0, \mu_x \neq 0 \\ \max\{d_x + d_y - 1/2, 0\}, & \text{for } \mu_y = \mu_x = 0 \text{ and } S_{xy} \neq 0 \\ d_x + d_y - 1/2, & \text{for } \mu_y = \mu_x = 0 \text{ and } S_{xy} = 0, \end{cases}$$

where $S_{xy} = \sum_{\tau=-\infty}^{\infty} \gamma_x(\tau)\gamma_y(\tau)$.

Proof. *The autocovariance function of any $x_t y_t$ satisfying Assumption 1 is given by (1). This is a linear combination of the autocovariance functions $\gamma_x(\tau)$, $\gamma_y(\tau)$, the cross-covariance function $\xi(\tau)$ and interaction terms between them. Since long memory is defined in (3) by the shape of the autocovariance function for $\tau \to \infty$, we can determine the memory of $x_t y_t$ by finding the limit of $\gamma_{xy}(\tau)$. For $\tau \to \infty$, we can substitute $\gamma_x(\tau)$ and $\gamma_y(\tau)$ with $G_x \tau^{2d_x - 1}$ and $G_y \tau^{2d_y - 1}$ from (3). The asymptotic properties of the cross-covariance function $\xi(\tau)$ can be derived from results of Phillips and Kim (2007). In Theorem 1, they show that the autocovariance matrix $\Gamma_{XX}(\tau)$ of a q-dimensional multivariate fractionally integrated process X_t is*

$$[\Gamma_{XX}(\tau)]_{ab} = \frac{2 f_{u_a u_b}(0) \Gamma(1 - d_a - d_b) \sin(\pi d_b)}{\tau^{1 - d_a - d_b}} + O\left(\frac{1}{\tau^{2 - d_a - d_b}}\right),$$

where A_{ab} denotes the element in the ath row and bth column of the matrix A. The asymptotic expansion of the Fourier integral used to derive this result is not specific to fractionally integrated processes, but holds for long memory processes in general. It therefore follows, that $\xi(\tau) = G_{xy} \tau^{d_a + d_b - 1} + o(\tau^{d_a + d_b - 1})$. Furthermore, since by Assumption 1 both x_t and y_t are causal, $\xi(-\tau) = 0$, so that the last term in (1) drops out.

Therefore, as $\tau \to \infty$, we have

$$\gamma_{xy}(\tau) = \mu_x^2 G_y \tau^{2d_y - 1} + \mu_y^2 G_x \tau^{2d_x - 1} + \mu_x \mu_y G_{xy} \tau^{d_x + d_y - 1} + G_x G_y \tau^{2(d_x + d_y - 1)} + o(\tau^{d_x + d_y - 1}).$$

Now, considering the exponents and setting $d_x + d_y - 1 = 2\bar{d}_3 - 1$ *and* $2(d_x + d_y - 1) = 2\bar{d}_4 - 1$ *gives* $\bar{d}_3 = (d_x + d_y)/2$ *and* $\bar{d}_4 = (d_x + d_y - 1/2)$*, so that*

$$\gamma_{xy}(\tau) = \mu_x^2 G_y \tau^{2d_y - 1} + \mu_y^2 G_x \tau^{2d_x - 1} + \mu_x \mu_y G_{xy} \tau^{2\bar{d}_3 - 1} + G_x G_y \tau^{2\bar{d}_4 - 1} + o(\tau^{d_x + d_y - 1}). \qquad (4)$$

Since $O(\tau^p) + O(\tau^q) = O(\tau^{\max(p,q)})$*, the autocovariance function* $\gamma_{xy}(\tau)$ *is dominated by the term with the largest memory parameter, as* $\tau \to \infty$*. The approximation error* $o(\tau^{d_x+d_y-1})$ *vanishes, because* $d_x, d_y < 1/2$*. Depending on the values of* μ_x *and* μ_y*, different cases can be distinguished.*

1. *If* $\mu_x = \mu_y = 0$*, (4) is reduced to* $\gamma_{xy}(\tau) \approx G_x G_y \tau^{2\bar{d}_4 - 1}$*. Therefore, the memory of* $x_t y_t$ *would be given by* $\bar{d}_4 = (d_x + d_y - 1/2)$*, which can be negative so that the decay rate of the autocovariance function is that of an antipersistent LM process. However, in this case the long memory definition is only fulfilled if the spectral density is zero at the origin, which is equivalent to* $S_{xy} = 0$*. Otherwise the process is* $LM(0)$*.*

2. *If* $\mu_x = 0 \neq \mu_y$*, (4) becomes* $\gamma_{xy}(\tau) \approx \mu_y^2 G_x \tau^{2d_x - 1} + G_x G_y \tau^{2\bar{d}_4 - 1}$ *and the dominating term is the maximum of* d_x *and* $\bar{d}_4 = (d_x + d_y - 1/2)$*. This is* d_x*, because* $d_y < 1/2$*.*

3. *If* $\mu_y = 0 \neq \mu_x$*, by the same arguments, the memory is* d_y*.*

4. *Finally, if* $\mu_x, \mu_y \neq 0$*, the memory order is the maximum of* d_x*,* d_y*,* $(d_x + d_y)/2$ *and* $d_x + d_y - 1/2$*. Furthermore, since* $d_x, d_y < 1/2$*,* $\max\{d_x, d_y\}$ *will always be at least as large as the other two terms.*

If $\rho_{xy} = 0$*, we have* $G_{xy} = 0$*, so that the third term in (4) is zero. However, this does not affect the memory properties of the product series compared to the case when* $\rho_{xy} \neq 0$*, because the memory of the third term is always dominated by that of the first two terms since* $\max\{d_x, d_y\} \geq (d_x + d_y)/2$.

□

It can immediately be seen from Proposition 1, that the means of x_t and y_t play a crucial role. If both means are non-zero, the asymptotic autocovariance function is dominated by the input series with the larger memory parameter. If both means are zero, the first three terms in (4) drop out, and the memory in z_t is $\bar{d}_4 = (d_x + d_y - 1/2)$. If one of the processes is mean zero, the memory is equal to that of this process, because $d_x, d_y > \bar{d}_4$.

Intuitively, if one thinks of long range dependence in terms of the persistence of deviations from the mean, it is obvious that this persistence vanishes if the series is multiplied with a zero mean *iid*-series. On the other hand, if the series is multiplied by a series with a non-zero mean, consecutive observations of the product series are likely to be located at the same side of the mean, so that the persistence remains intact.

Since the decay of the autocovariances of antipersistent LM-processes is initially very fast, the condition $S_{xy} = 0$ is basically a condition on the first autocovariances $\gamma_{xy}(\tau)$. As can be seen from (1), this is equal to the product $\gamma_{xy}(\tau) = \gamma_x(\tau)\gamma_y(\tau)$, if $\mu_x = \mu_y = 0$. The process can therefore only be LM with negative d, if $\gamma_x(\tau)$ and $\gamma_y(\tau)$ have different signs and the sum of their product over the first leads and lags is close to the process variance. If $\gamma_{xy}(\tau)$ is negative but $S_{xy} \neq 0$, the process has antipersistent short memory. Finally, if $\gamma_{xy}(\tau)$ is positive, the process is simply $LM(0)$ if $d_x + d_y - 1/2$ is negative.[2]

3 Memory of Squared Series

An important special case of Proposition 1 arises if $x_t = y_t$, so that the product becomes the square of one series. This gives the following corollary.

Corollary 1. *For x_t satisfying Assumption 1, the square $z_t = x_t^2$ is $LM(d_z)$ with*

$$d_z = \begin{cases} d_x, & \text{if } \mu_x \neq 0 \\ \max\{2d_x - 1/2, 0\}, & \text{if } \mu_x = 0. \end{cases}$$

Proof. *Wecker (1978) shows that for x_t^2 equation (1) simplifies to*

$$\gamma_{xx}(\tau) = 4\mu_x^2\gamma_x(\tau) + 2\gamma_x^2(\tau).$$

For $\tau \to \infty$, this gives

$$\gamma_{xx}(\tau) \approx 4\mu_x^2 G_x \tau^{2d_x - 1} + 2G_x\tau^{2(2d_x-1)}. \tag{5}$$

Again, from equating $2(2d_x - 1) = 2d_{sq} - 1$ we have $d_{sq} = 2d_x - 1/2$ so that $\gamma_{xx}(\tau) \approx 4\mu_x^2 G_x\tau^{2d_x-1} + 2G_x\tau^{2d_{sq}-1}$. Since $d_x < 1/2$, by Assumption 1, $d_{sq} < d_x$ so that the first term in (5) always dominates the second if $\mu_x \neq 0$. For $\mu_x = 0$, the memory is determined by the second

[2] A similar issue arises in Dittmann and Granger (2002). However, they consider pure fractionally integrated processes and do not allow for short run dynamics. In this case all autocovariances are positive and the lower bound for the memory of the transformed series is zero.

term. This is $\max\{d_{sq}, 0\}$*, for the reasons discussed in the proof of Proposition 1, above. Since* $\gamma_x(\tau)^2 \geq 0$*, the process cannot become antipersistent and the lower bound for the memory order of the square is always zero.* $\square$

Corollary 1 shows, that the memory of a squared series will be reduced if the series is mean zero, whereas the memory is unaffected by this non-linear transformation if it has a non-zero mean. In case of a reduction, the memory will be zero, for $d_x \leq 0.25$.

The memory of the squared zero mean series is also a special case of the results in Dittmann and Granger (2002). As discussed above, they derive the properties of non-linear transformations of fractionally integrated time series with zero mean and unit variance under the restriction that the non-linear transformation can be expressed as a finite sum of Hermite polynomials. Corollary 1 shows that the mean zero assumption is critical, since squaring the series does not cause a reduction in memory if it is not fulfilled.

Note that it is not possible to apply Proposition 1 together with Corollary 1 to determine the memory of higher order power transformations such as x_t^3, because the Gaussianity requirement in Assumption 1 is no longer satisfied by the squared series x_t^2.

4 Products of Fractionally Cointegrated Time Series

Another interesting special case is the product of series with a common factor, such as fractionally cointegrated series. Consider the following model

$$x_t = \beta_x + \delta_x u_t + \eta_t \tag{6}$$

$$y_t = \beta_y + \delta_y u_t + \varepsilon_t, \tag{7}$$

where u_t is $LM(d_u)$, $\eta_t \sim LM(d_\eta)$ and $\varepsilon_t \sim LM(d_\varepsilon)$ fulfill the conditions imposed in Assumption 1, and β_x and β_y are finite real constants. Furthermore, let η_t and ε_t be mean zero, with $d_\eta = d_u - b \geq 0$ and $d_\varepsilon = d_u - b - \epsilon \geq 0$, for some constants $b, \epsilon \geq 0$. Clearly, both y_t and x_t are driven by the common long range dependent factor u_t and they are both $LM(d_u)$. We therefore refer to them as series with common long memory. As before $0 \leq d_u < 0.5$ and of course δ_x, $\delta_y \neq 0$. We then obtain the following result.

Proposition 2. *Let* y_t *and* x_t *have common long memory of order* d_u*, so that they can be represented as in equations (6) and (7). Then the product* $x_t y_t$ *is* $LM(d_u)$ *if either* $(\beta_x\delta_y + \beta_y\delta_x)$

or μ_u *is unequal zero and it is* $LM(\max\{2d_u - 1/2, 0\})$ *if* $\mu_x = \mu_y = \mu_u = 0$.

Proof. *By substituting the relationships in (6) and (7) and rearranging the terms, the product* $x_t y_t$ *is given by*

$$x_t y_t = \underbrace{\beta_x \beta_y}_{I} + \underbrace{(\beta_x \delta_y + \beta_y \delta_x) u_t}_{II} + \underbrace{\delta_x \delta_y u_t^2}_{III} + \underbrace{(\beta_x + \delta_x u_t)\varepsilon_t}_{IV} + \underbrace{(\beta_y + \delta_y u_t)\eta_t}_{V} + \underbrace{\eta_t \varepsilon_t}_{VI}. \quad (8)$$

Proposition 3 in Chambers (1998) states that a linear combination of fractionally integrated processes is itself fractionally integrated, with an order of integration equal to the maximum of those in the linear combination. Chamber's arguments extend readily to long memory processes in general because they only make use of the long memory properties. More specifically, let $f_X(\lambda)$ *be the spectral density matrix of the multivariate long memory process* X_t *that fulfills* $[Re(f_X(\lambda))]_{ab} \sim g_{ab}|\lambda|^{-d_a - d_b}$, *as* $\lambda \to 0$. *Than the linear combination* $S_t = w' X_t$ *has spectral density*

$$f_S(\lambda) = w' f_X(\lambda) w \sim \sum_{a=1}^{q} w_a^2 G_{aa} |\lambda|^{-2d_a} + \sum_{a \neq b} w_a w_b G_{ab} |\lambda|^{-d_a - d_b}, \quad (9)$$

as $\lambda \to 0$. *Since* $\mathcal{O}(|\lambda|^{-2d_a}) + \mathcal{O}(|\lambda|^{-2d_b}) = \mathcal{O}(|\lambda|^{-2\max(d_a, d_b)})$, *as* $\lambda \to 0$, *and* $2\max(d_a, d_b) > d_a + d_b$, *the spectral density is proportionate to the largest term in the first sum on the right hand side of (9).*

The memory order of $x_t y_t$ *is therefore the maximum of the memory orders in terms I to VI in (8) and the memory of these individual terms can be determined from the results in Proposition 1 and Corollary 1.*

In Proposition 2, two cases are distinguished. In the first one, either μ_u *or* $\beta_x \delta_y + \beta_y \delta_x$ *are unequal zero. In the second,* $\mu_x = \mu_y = \mu_u = 0$. *These cases are considered separately.*

1. *Whenever* $\mu_u \neq 0$, *III is* $LM(d_u)$, *by Corollary 1. Since we know from Proposition 1 that none of the other terms can have stronger memory than the original series, it follows directly from the result of Chambers (1998) that* $x_t y_t$ *will be* $LM(d_u)$. *The same holds true if* $\beta_x \delta_y + \beta_y \delta_x \neq 0$, *since* u_t *is* $LM(d_u)$.

2. *If* $\mu_x = \mu_y = \mu_u = 0$, β_x *and* β_y *in II are zero from (6) and (7) and term III is* $LM(\max\{2d_u - 1/2, 0\})$, *by Corollary 1. From Proposition 3 in Chambers (1998) the lower bound for the memory of the linear combination in (8) is therefore zero. By Proposition*

1, the reduced memory of the terms IV, V and VI is $d_u + d_\varepsilon - 1/2$, $d_u + d_\eta - 1/2$ and $d_u + d_\varepsilon - 1/2$, respectively. Since $d_\varepsilon, d_\eta < d_u$, by definition, all of these terms are smaller than $2d_u - 1/2$. The memory of $x_t y_t$ is therefore determined by that of term III. $\square$

When comparing Propositions 1 and 2, one can see that the memory in the product series is less fugacious if the factor series have common long memory. As before, the nature of the transmission depends on the means of the series. However, since the square in term III determines the memory of $x_t y_t$ in equation (8), it is now the mean of the common factor that is crucial. Therefore, z_t can be d_u, even if $\mu_x = \mu_y = 0$.

5 Conclusion

This paper derives the memory of products of long memory time series. It is found that the nature of the transmission critically hinges on the means of the factor series. While the memory in the product series will be reduced if the means are zero, the memory of the more persistent factor series will be directly propagated if the factor series have non-zero means. These findings show that the property of long range dependence is less fugacious than it might seem from previous results on non-linear transformations of long memory series obtained by Dittmann and Granger (2002) that rely on a mean zero assumption.

One may conjecture that the transmission of long memory to other non-linear transformations may similarly depend on the means. Further research on the memory properties of a broader class of non-linear transformations would therefore be of great interest.

References

Beran, J., Feng, Y., Ghosh, S., and Kulik, R. (2013). *Long memory processes: Probabilistic properties and statistical methods*. Springer London, Limited.

Bohrnstedt, G. W. and Goldberger, A. S. (1969). On the exact covariance of products of random variables. *Journal of the American Statistical Association*, 64(328):1439–1442.

Chambers, M. J. (1998). Long memory and aggregation in macroeconomic time series. *International Economic Review*, 39(4):1053–1072.

Corradi, V. (1995). Nonlinear transformations of integrated time series: a reconsideration. *Journal of Time Series Analysis*, 16(6):539–549.

Dittmann, I. and Granger, C. W. (2002). Properties of nonlinear transformations of fractionally integrated processes. *Journal of Econometrics*, 110(2):113–133.

Goodman, L. A. (1960). On the exact variance of products. *Journal of the American Statistical Association*, 55(292):708–713.

Granger, C. W. and Hallman, J. (1991). Nonlinear transformations of integrated time series. *Journal of Time Series Analysis*, 12(3):207–224.

Phillips, P. C. B. and Kim, C. S. (2007). Long-run covariance matrices for fractionally integrated processes. *Econometric Theory*, 23(6):1233–1247.

Wecker, W. E. (1978). A note on the time series which is the product of two stationary time series. *Stochastic Processes and their Applications*, 8(2):153–157.