Fixed-Bandwidth CUSUM Tests Under Long Memory

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December 7, 2018

Abstract

We propose a family of self-normalized CUSUM tests for structural change under long memory. The test statistics apply non-parametric kernel-based fixed-b and fixed-m long-run variance estimators and have well-defined limiting distributions that only depend on the long-memory parameter. A Monte Carlo simulation shows that these tests provide finite sample size control while outperforming competing procedures in terms of power.

Key words: Fixed-bandwidth asymptotics · Fractional Integration · Long Memory · Structural Breaks.

JEL classification: C12, C22

1 Introduction

Distinguishing structural change from long memory is notoriously difficult. Both features not only generate similar empirical behavior, such as a singularity in the spectrum at the zero frequency and significant autocorrelations at large lags, but the presence of long memory also decreases the rate of convergence for estimates of the mean. Mean changes are therefore hard to identify.

The testing problem is addressed by several papers such as Wang (2008), Shao (2011), Dehling, Rooch, and Taqqu (2013), Iacone, Leybourne, and Taylor (2014), Betken (2016), and others, who employ a variety of competing testing principles. These are reviewed in Wenger, Leschinski, and Sibbertsen (2018). It is found that self-normalized tests are robust against size distortions in finite samples and the CUSUM testing principle tends to be the most powerful. Overall the sup-Wald fixed-b test by Iacone, Leybourne, and Taylor (2014) and the self-normalized Wilcoxon-type test by Betken (2016) are found to offer the best trade-off between size and power.

These findings motivate our contribution to combine the CUSUM testing principle with fixed-bandwidth estimators for the long-run variance. To estimate the long-run variance, we apply both the standard fixed-b approach following Kiefer and Vogelsang (2005) and McElroy and Politis (2012) as well as the newly proposed frequency domain analogue of Hualde and Iacone (2017) that is referred to as the "fixed-m" approach. For the calculation of the autocovariances and the periodogram that these statistics rely on, we employ two different centering approaches: standard demeaning and an alternative demeaning approach that accounts for the possibility of a

mean shift under the alternative.
In an extensive Monte Carlo simulation we examine the finite sample size and power properties of our tests in various settings. We further compare them with the sup-Wald fixed-b test by Iacone, Leybourne, and Taylor (2014) and the self-normalized Wilcoxon-type test by Betken (2016) and find that our tests outperform these competitors in terms of power, while maintaining comparable finite sample size properties.
The rest of the paper is structured as follows. Section 2 describes the model, its assumptions and provides and discusses our new CUSUM fixed-bandwidth tests. In Section 3, a Monte Carlo simulation is conducted and we compare our tests with existing procedures. Section 4 concludes.

2 Fixed-Bandwidth CUSUM Tests

We consider a single mean shift model where the stochastic process $(y_t)_{t\geq 1}$ is generated by

$$y_t = \mu + \beta \mathbb{1}(t \geq t^*) + \epsilon_t, \qquad \text{for} \qquad t = 1, ..., T. \tag{1}$$

Here, $\mu_t = \mu + \beta \mathbb{1}(t \geq t^*)$ for $t \geq 0$ is a deterministic sequence of regression means that fulfills $|\mu_t| < \infty$. The coefficient β determines the magnitude of the possible mean shift, $\mathbb{1}(t \geq t^*)$ is an indicator function that depends on the location of the break point denoted by $t^* = \lfloor \tau^* T \rfloor$, where $\lfloor \cdot \rfloor$ returns the integer part of its argument, and $\tau^* \in (0,1)$ denotes the break fraction. For the long-memory error term $(\epsilon_t)_{t\geq 1}$ the following assumption holds.

Assumption 1. *For $T \to \infty$ there exists some $|d| < 1/2$ such that*

$$T^{-1/2-d} \sum_{t=1}^{\lfloor sT \rfloor} \epsilon_t \Rightarrow \sigma B_d(s) \quad \textit{for} \quad 0 \leq s \leq 1, \tag{2}$$

where $B_d(s)$ is a (standard) fractional Brownian motion of type I and "$\Rightarrow$" denotes convergence in distribution in the Skorokhod space $\mathcal{D}[0,1]$.

Our interest lies on testing the null hypothesis of no mean shift

$$H_0 : \mu_1 = ... = \mu_T = \mu$$

against the alternative of a single mean shift

$$H_1 : \mu_t \neq \mu_s \quad \text{for some} \quad 1 < t, s < T.$$

The CUSUM test for a change-in-mean is given by

$$Z = \sup_{\tau \in [\tau_1, \tau_2]} = \left| \frac{1}{T^{1/2}\sigma} \sum_{t=1}^{\lfloor \tau T \rfloor} \widehat{\epsilon}_t \right|, \tag{3}$$

with $\widehat{\epsilon}_t = y_t - \bar{y}$ and where σ^2 is the long-run variance. The search area in which the test statistic is calculated is restricted to $0 < \tau_1 < \tau_2 < 1$. For the estimation of the long-run variance, we consider fixed-b estimators as introduced by Kiefer and Vogelsang (2005) and extended to long-memory processes by McElroy and Politis (2012).

For the calculation of the empirical autocovariances we consider two alternative demeaning procedures: type-I demeaning that assumes a constant mean ($\beta = 0$) and type-II demeaning that allows for the possibility of a non-zero mean shift ($\beta \neq 0$) at sample fraction τ. Hence, with a type-I estimator the autocovariances are estimated under H_0 and with a type-II estimator and the correct choice of τ the autocovariances are estimated under H_1.

The resulting fixed-b long-run variance estimators are given by

$$\widehat{\sigma}^2_{I,b} = \widehat{\gamma}^I_0 + 2\sum_{j=1}^{T-1} k\left(\frac{j}{B}\right)\widehat{\gamma}^I_j \,, \tag{4}$$

$$\text{and} \qquad \widehat{\sigma}^2_{II,b}(\tau) = \widehat{\gamma}^{II}_0(\tau) + 2\sum_{j=1}^{T-1} k\left(\frac{j}{B}\right)\widehat{\gamma}^{II}_j(\tau) \,, \tag{5}$$

where $k(\cdot)$ is the user-chosen kernel function with bandwidth parameter B. Furthermore, $\widehat{\gamma}^I_j = \frac{1}{T}\sum_{t=|j|+1}^{T}\widehat{\epsilon}^I_t\widehat{\epsilon}^I_{t-|j|}$ and $\widehat{\gamma}^{II}_j = \frac{1}{T}\sum_{t=|j|+1}^{T}\widehat{\epsilon}^{II}_t(\tau)\widehat{\epsilon}^{II}_{t-|j|}(\tau)$, with

$$\begin{aligned}
\widehat{\epsilon}^I_t &= y_t - \bar{y}, \\
\widehat{\epsilon}^{II}_t(\tau) &= y_t - I(t \leq \lfloor \tau T \rfloor)\bar{y}(\tau) - I(t > \lfloor \tau T \rfloor)\underline{y}(\tau),
\end{aligned}$$

$\bar{y}(\tau) = \lfloor \tau T \rfloor^{-1}\sum_{t=1}^{\lfloor \tau T \rfloor} y_t$, and $\underline{y}(\tau) = \lfloor (1-\tau)T \rfloor^{-1}\sum_{t=\lfloor \tau T \rfloor}^{T} y_t$. Here and in the following, we employ the commonly used Bartlett kernel given by $k = 1 - |x|$ for $|x| \leq 1$ and $k(x) = 0$, otherwise. $\bar{y}(\tau)$ and $\underline{y}(\tau)$ are the means of the first and second part of the sample that is split at sample fraction τ. Under the null hypothesis of no structural change both are consistent estimators for μ. Under the alternative, $\bar{y}(\tau^*)$ is consistent for μ and $\underline{y}(\tau^*)$ for $\mu + \beta$. Consequently, the demeaned process $\widehat{\epsilon}^{II}_t(\tau^*)$ is mean zero for all $t = 1, ..., T$.

For $B/T \to 0$ as $T \to \infty$ the estimators in (4) and (5) are standard HAC estimators. In contrast to that, the fixed-b framework proposed by Kiefer and Vogelsang (2005) provides an alternative asymptotic theory, where it es assumed that B/T approaches a fixed constant $b \in (0, 1]$ as $T \to \infty$. That means that the bandwidth parameter B is selected such that an asymptotically non-zero fraction of the empirical autocovariances is considered. This has the consequence that the estimation is no longer consistent. The fixed-b estimator $\widehat{\sigma}^2_b$ converges to σ^2 multiplied by a functional of a Brownian bridge process that depends on the kernel and b. In contrast to test statistics using standard HAC estimators that are often size distorted in finite samples, the resulting fixed-b tests have non-standard limit distributions, but better finite sample size control. The type-II demeaning is motivated by the fact that the mean shift under the alternative introduces spurious autocorrelation. Equation (5) in Mikosch and Stărică (2004) implies for our mean-shift model in Equation (1) that the empirical autocovariance function at any fixed lag j fulfills

$$\widehat{\gamma}^I_j \to \gamma_j + \tau^*(1-\tau^*)\beta^2, \tag{6}$$

as $T \to \infty$. This means that the autocovariance function is biased upwards by a constant that depends on the size and location of the mean shift. Since the long-run variance estimator $\widehat{\sigma}^2_{I,b}$ in Equation (4) is a weighted average of all available empirical autocovariances, this introduces a positive bias under the alternative hypothesis. This bias is increasing in b and since the long-run variance estimate is used for the standardization of the centered partial sum of the process, this leads to a considerable power loss for larger values of b.

The partial sum of the type-II demeaned process $\widehat{\epsilon}_t^{II}(\tau)$ fulfills the following FCLT.

Lemma 1. *For* $0 < s, \tau < 1$ *and under Assumption 1 we have*

$$T^{-1/2-d} \sum_{t=1}^{\lfloor sT \rfloor} \widehat{\epsilon}_t^{II}(\tau) \Rightarrow \sigma \widetilde{B}_d^{II}(s,\tau)$$

$$\text{where} \qquad \widetilde{B}_d^{II}(s,\tau) := B_d(s) - \frac{min(s,\tau)}{\tau} B_d(\tau) - \frac{max(s-\tau,0)}{1-\tau}\left(B_d(1) - B_d(\tau)\right).$$

The proof of this and all following results can be found in the appendix. Lemma 1 shows that the partial sum of the process $\widehat{\epsilon}_t^{II}(\tau)$ demeaned with $\bar{y}(\tau)$ and $\underline{y}(\tau)$ converges to a generalized version of a fractional Brownian bridge that passes through zero at $s = \tau$. We refer to this as a fractional Brownian bridge of type II.

As an alternative fixed-bandwidth approach Hualde and Iacone (2017) propose a frequency-based normalization that utilizes the first m periodogram ordinates local to the zero frequency and assumes the bandwidth m to be fixed as $T \to \infty$. Depending on the type of demeaning this so-called fixed-m estimator is then given by

$$\widehat{\sigma}^2_{I,m} = \frac{2\pi}{m} \sum_{j=1}^{m} I^I(\lambda_j), \tag{7}$$

$$\text{and} \qquad \widehat{\sigma}^2_{II,m}(\tau) = \frac{2\pi}{m} \sum_{j=1}^{m} I^{II}(\lambda_j,\tau), \tag{8}$$

where $I^I(\lambda) = \left|(2\pi T)^{-1/2} \sum_{t=1}^{T} \widehat{\epsilon}_t^I e^{i\lambda t}\right|^2$ and $I^{II}(\lambda,\tau) = \left|(2\pi T)^{-1/2} \sum_{t=1}^{T} \widehat{\epsilon}_t^{II}(\tau) e^{i\lambda t}\right|^2$ are the periodograms of the respective demeaned processes evaluated at the Fourier frequencies $\lambda_j = 2\pi j/T$. Again, the estimator does not converge to the true long-run variance, but to σ multiplied by a functional of a (fractional) Brownian bridge.

The family of test statistics proposed here is obtained by combining the CUSUM testing principle in (3), with the type-I and type-II fixed-b and fixed-m normalizations in Equations (4) to (5) and (7) to (8). These are given by

$$Z_\delta^I := \sup_{\tau \in [\tau_1,\tau_2]} \left| \frac{1}{T^{1/2}\widehat{\sigma}_{I,\delta}} \sum_{t=1}^{\lfloor \tau T \rfloor} \widehat{\epsilon}_t^I \right|, \tag{9}$$

$$Z_\delta^{II} := \sup_{\tau \in [\tau_1,\tau_2]} \left| \frac{1}{T^{1/2}\widehat{\sigma}_{II,\delta}(\tau)} \sum_{t=1}^{\lfloor \tau T \rfloor} \widehat{\epsilon}_t^I \right|, \tag{10}$$

where $\delta \in \{b, m\}$.

The limiting distributions of these statistics are established in the following theorem.

Theorem 1. *Under Assumption 1 we have under H_0:*

$$Z_\delta^I \xrightarrow{d} \sup_{\tau\in[\tau_1,\tau_2]} \left|\frac{\widetilde{B}_d^I(\tau)}{\sqrt{Q_{I,\delta}}}\right|,$$
$$Z_\delta^{II} \xrightarrow{d} \sup_{\tau\in[\tau_1,\tau_2]} \left|\frac{\widetilde{B}_d^I(\tau)}{\sqrt{Q_{II,\delta}(\tau)}}\right|,$$

where

$$Q_{I,b} = \frac{2}{b}\left(\int_0^1 \widetilde{B}_d^I(r)^2 \mathrm{d}r - \int_0^{1-b} \widetilde{B}_d^I(r+b)\widetilde{B}_d^I(r)\mathrm{d}r\right)$$
$$Q_{I,m} = \frac{1}{m}\sum_{j=1}^m \left\{\left(2\pi j\int_0^1 \sin(2\pi jr)\widetilde{B}_d^I(r)dr\right)^2 + \left(2\pi j\int_0^1 \cos(2\pi jr)\widetilde{B}_d^I(r)dr\right)^2\right\}$$
$$Q_{II,b}(\tau) = \frac{2}{b}\left(\int_0^1 \widetilde{B}_d^{II}(r,\tau)^2 \mathrm{d}r - \int_0^{1-b} \widetilde{B}_d^{II}(r+b,\tau)\widetilde{B}_d^{II}(r,\tau)\mathrm{d}r\right)$$
$$Q_{II,m}(\tau) = \frac{1}{m}\sum_{j=1}^m \left\{\left(2\pi j\int_0^1 \sin(2\pi jr)\widetilde{B}_d^{II}(r,\tau)dr\right)^2 + \left(2\pi j\int_0^1 \cos(2\pi jr)\widetilde{B}_d^{II}(r,\tau)dr\right)^2\right\}.$$

Here $\widetilde{B}_d^I(s) = B_d(s) - sB_d(1)$ is a standard fractional Brownian bridge and $\widetilde{B}_d^{II}(s,\tau)$ is a fractional Brownian bridge of type II as defined in Lemma 1. Convergence in distribution is denoted by "$\xrightarrow{d}$".

The limiting distributions of the test statistics depend on the long-memory parameter d and reflect the user choices of the bandwidth, and the search area $[\tau_1, \tau_2]$. Asymptotic critical values can be found in Tables 5 to 8.

3 Simulation Studies

To examine the finite sample size and power properties of the proposed test statistics, we conduct a Monte Carlo simulation study. We consider an ARFIMA(1,d,0) process for ϵ_t and choose $d \in \{0; 0.2; 0.4\}$, AR coefficients $\phi \in \{0; 0.5\}$, and sample sizes $T \in \{250; 500; 1000; 2000\}$. To estimate d, we apply the local Whittle estimator using a bandwidth of $\lfloor T^{0.8} \rfloor$. The nominal significance level is $\alpha = 5\%$, the search area is $[\tau_1, \tau_2] = [0.15, 0.85]$, and we report the rejection frequencies obtained from $M = 1,000$ replications.

The finite sample size results of the CUSUM fixed-b and CUSUM fixed-m tests of type I and II respectively are given on the left-hand side of Tables 1 to 4.

The CUSUM fixed-b type-I test roughly holds its nominal size for all d and all sample sizes T when there are no short-run dynamics. If $\phi = 0.5$, it also holds its nominal size, except for $b \geq 0.5$ where it is oversized even at large sample sizes. In contrast, the type-II test does not exceed its nominal size, but it is conservative if $\phi = 0.5$.

The CUSUM fixed-m tests show a similar pattern. For the CUSUM fixed-m type-I test, we observe that it does not exceed its size at all sample sizes T, bandwidths m, and d when $\phi = 0$. If we have short-run dynamics in the sample, it is slightly oversized for small m and also slightly undersized for large m. The CUSUM fixed-m type-II test has a better size control at all

	CUSUM fixed-b type I											
	size						power					
	$\phi = 0$			$\phi = 0.5$			$\phi = 0$			$\phi = 0.5$		
T/d	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4
	b=0.05											
250	0.036	0.053	0.048	0.001	0.004	0.013	1.000	0.778	0.214	0.417	0.113	0.110
500	0.036	0.041	0.043	0.004	0.003	0.011	1.000	0.964	0.346	0.939	0.271	0.159
1000	0.050	0.049	0.058	0.000	0.003	0.028	1.000	0.996	0.446	1.000	0.554	0.220
2000	0.049	0.053	0.036	0.008	0.002	0.028	1.000	1.000	0.523	1.000	0.854	0.280
	b=0.1											
250	0.047	0.048	0.043	0.016	0.018	0.042	0.996	0.686	0.247	0.668	0.244	0.162
500	0.046	0.043	0.046	0.022	0.029	0.043	1.000	0.896	0.328	0.972	0.468	0.220
1000	0.054	0.053	0.046	0.036	0.025	0.045	1.000	0.976	0.435	1.000	0.718	0.274
2000	0.058	0.060	0.055	0.032	0.038	0.038	1.000	0.999	0.504	1.000	0.928	0.342
	b=0.2											
250	0.043	0.036	0.043	0.051	0.067	0.048	0.501	0.313	0.129	0.444	0.246	0.104
500	0.044	0.038	0.059	0.069	0.061	0.058	0.689	0.456	0.175	0.628	0.349	0.124
1000	0.071	0.042	0.039	0.083	0.066	0.061	0.832	0.565	0.207	0.798	0.469	0.185
2000	0.069	0.050	0.059	0.065	0.072	0.050	0.949	0.692	0.257	0.893	0.596	0.203
	b=0.5											
250	0.037	0.035	0.038	0.143	0.091	0.054	0.000	0.001	0.016	0.000	0.020	0.022
500	0.047	0.043	0.046	0.113	0.101	0.059	0.000	0.001	0.011	0.000	0.006	0.016
1000	0.055	0.055	0.049	0.128	0.100	0.060	0.000	0.000	0.011	0.000	0.006	0.020
2000	0.054	0.048	0.061	0.124	0.138	0.051	0.000	0.000	0.011	0.000	0.000	0.025
	b=0.8											
250	0.030	0.038	0.036	0.139	0.091	0.054	0.000	0.003	0.018	0.000	0.020	0.032
500	0.037	0.023	0.045	0.153	0.108	0.054	0.000	0.001	0.019	0.000	0.008	0.032
1000	0.051	0.044	0.048	0.146	0.120	0.079	0.000	0.000	0.014	0.000	0.003	0.029
2000	0.073	0.054	0.049	0.142	0.117	0.073	0.000	0.000	0.013	0.000	0.001	0.024
	b=1											
250	0.034	0.038	0.035	0.131	0.113	0.055	0.000	0.003	0.028	0.000	0.027	0.033
500	0.038	0.046	0.055	0.141	0.139	0.047	0.000	0.001	0.013	0.000	0.006	0.029
1000	0.051	0.045	0.037	0.126	0.121	0.066	0.000	0.000	0.023	0.000	0.005	0.024
2000	0.049	0.047	0.053	0.146	0.118	0.055	0.000	0.000	0.009	0.000	0.001	0.022

Table 1: **Size and power results of the CUSUM fixed-b type-I test** at different sample sizes T and different long-memory parameters d applying the local Whittle estimator. The number of replications is $1,000$.

specifications.

The finite sample power results are given on the right hand side of Tables 1 to 4. For the power simulations, we add a single mean shift in the middle of the sample which is as large as the standard deviation of the time series. Again, the power results of the fixed-b and fixed-m CUSUM tests are quite similar.

The CUSUM fixed-b type-I test does not have power at all for $b \geq 0.5$ for the reasons discussed in the context of Equation (6). The estimated long-run variance is too high because the mean shift in the series generates high autocovariances at large lags which are used in the long-run variance estimation when b is large. For small b however, the test has good power properties — especially with $b = 0.1$ (which is the same bandwidth Iacone, Leybourne, and Taylor (2014) choose for their sup-Wald fixed-b test).

In general, with increasing T the power increases and with increasing ϕ the power decreases. The power also decreases with increasing d as usual for tests on a mean shift in long-memory

	CUSUM fixed-b type II											
	size						power					
	$\phi = 0$			$\phi = 0.5$			$\phi = 0$			$\phi = 0.5$		
T/d	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4
	b=0.05											
250	0.033	0.044	0.054	0.001	0.000	0.017	1.000	0.789	0.278	0.425	0.123	0.147
500	0.055	0.041	0.045	0.001	0.003	0.026	1.000	0.947	0.323	0.926	0.232	0.164
1000	0.051	0.042	0.050	0.002	0.008	0.023	1.000	1.000	0.417	1.000	0.495	0.203
2000	0.050	0.048	0.051	0.004	0.010	0.020	1.000	1.000	0.465	1.000	0.820	0.234
	b=0.1											
250	0.058	0.054	0.038	0.005	0.002	0.024	1.000	0.730	0.213	0.462	0.137	0.122
500	0.053	0.045	0.048	0.001	0.010	0.025	1.000	0.920	0.318	0.922	0.236	0.142
1000	0.055	0.053	0.046	0.006	0.007	0.030	1.000	0.987	0.326	1.000	0.478	0.174
2000	0.048	0.066	0.043	0.005	0.009	0.027	1.000	1.000	0.436	1.000	0.757	0.212
	b=0.2											
250	0.038	0.042	0.053	0.002	0.009	0.026	0.997	0.633	0.202	0.488	0.168	0.101
500	0.053	0.046	0.035	0.007	0.009	0.031	1.000	0.834	0.254	0.865	0.291	0.150
1000	0.057	0.062	0.052	0.009	0.008	0.036	1.000	0.954	0.344	0.996	0.469	0.176
2000	0.059	0.041	0.038	0.009	0.008	0.040	1.000	0.989	0.370	1.000	0.728	0.227
	b=0.5											
250	0.044	0.051	0.042	0.005	0.013	0.042	0.990	0.613	0.229	0.493	0.183	0.119
500	0.037	0.036	0.042	0.006	0.011	0.038	1.000	0.793	0.259	0.839	0.290	0.164
1000	0.055	0.051	0.050	0.003	0.013	0.045	1.000	0.905	0.319	0.995	0.497	0.195
2000	0.050	0.063	0.042	0.008	0.015	0.034	1.000	0.983	0.383	1.000	0.704	0.210
	b=0.8											
250	0.040	0.043	0.045	0.006	0.013	0.029	0.982	0.613	0.218	0.498	0.155	0.116
500	0.053	0.047	0.053	0.011	0.011	0.030	0.999	0.787	0.269	0.863	0.262	0.138
1000	0.044	0.052	0.055	0.002	0.014	0.035	1.000	0.928	0.270	0.985	0.435	0.153
2000	0.043	0.054	0.044	0.010	0.009	0.038	1.000	0.980	0.347	1.000	0.657	0.212
	b=1											
250	0.041	0.050	0.044	0.005	0.017	0.050	0.995	0.661	0.212	0.528	0.190	0.124
500	0.060	0.053	0.050	0.007	0.015	0.038	1.000	0.791	0.234	0.863	0.274	0.129
1000	0.049	0.043	0.053	0.010	0.016	0.038	1.000	0.927	0.294	0.989	0.496	0.179
2000	0.047	0.060	0.067	0.018	0.019	0.029	1.000	0.981	0.355	1.000	0.699	0.191

Table 2: **Size and power results of the CUSUM fixed-b type-II test** at different sample sizes T and different long-memory parameters d applying the local Whittle estimator. The number of replications is $1,000$.

time series. This is due to the fact that it is more difficult to observe a mean shift of fixed size in a more persistent time series.

As intended, the fixed-b type-II test which accounts for the mean shift in the long-run variance estimation generates power for all bandwidths b. Apart from that, it shows the same pattern as the type-I test and we suggest using $b = 0.1$.

When we compare both the type-I and type-II version of the fixed-b test, we observe that the type-II test has higher power than the type-I test for small d, but lower power in most cases for larger d and when short-run dynamics are present.

In case of the CUSUM fixed-m type-I test, we observe that it does not generate power for very small m, which is similar to the CUSUM fixed-b test of type I with large values of b. For larger m its power increases as the sample size increases and its power decreases when d increases. If we compare the setting with and without short-run dynamics, we observe that the power decreases as ϕ increases. The higher the bandwidth m, the higher is the power of the test when $\phi = 0$.

	CUSUM fixed-m type I											
	size						power					
	$\phi = 0$			$\phi = 0.5$			$\phi = 0$			$\phi = 0.5$		
T/d	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4
	m=1											
250	0.044	0.061	0.057	0.109	0.079	0.054	0.000	0.000	0.012	0.000	0.014	0.027
500	0.040	0.055	0.059	0.089	0.079	0.051	0.000	0.000	0.010	0.000	0.003	0.025
1000	0.043	0.050	0.054	0.099	0.086	0.064	0.000	0.000	0.008	0.000	0.000	0.023
2000	0.054	0.053	0.052	0.081	0.092	0.067	0.000	0.000	0.003	0.000	0.000	0.016
	m=2											
250	0.040	0.052	0.049	0.111	0.078	0.056	0.010	0.056	0.067	0.167	0.098	0.067
500	0.047	0.032	0.046	0.106	0.079	0.050	0.001	0.049	0.082	0.106	0.106	0.067
1000	0.044	0.063	0.044	0.120	0.091	0.051	0.000	0.029	0.072	0.032	0.153	0.065
2000	0.060	0.065	0.063	0.123	0.095	0.067	0.000	0.009	0.088	0.004	0.124	0.094
	m=3											
250	0.038	0.041	0.050	0.062	0.054	0.054	0.556	0.286	0.129	0.404	0.189	0.090
500	0.051	0.042	0.042	0.084	0.063	0.046	0.724	0.381	0.161	0.606	0.286	0.123
1000	0.049	0.040	0.050	0.078	0.061	0.045	0.881	0.509	0.167	0.777	0.405	0.118
2000	0.059	0.054	0.049	0.092	0.058	0.056	0.977	0.624	0.190	0.897	0.520	0.167
	m=4											
250	0.042	0.047	0.060	0.029	0.024	0.037	0.977	0.631	0.248	0.573	0.236	0.138
500	0.046	0.039	0.048	0.050	0.037	0.037	1.000	0.812	0.291	0.890	0.388	0.152
1000	0.048	0.066	0.053	0.058	0.041	0.028	1.000	0.935	0.340	0.992	0.590	0.202
2000	0.047	0.049	0.042	0.062	0.047	0.034	1.000	0.990	0.405	1.000	0.785	0.239
	m=10											
250	0.053	0.042	0.043	0.003	0.002	0.018	1.000	0.783	0.311	0.481	0.171	0.142
500	0.044	0.036	0.043	0.005	0.003	0.023	1.000	0.968	0.365	0.937	0.289	0.208
1000	0.045	0.044	0.050	0.007	0.004	0.032	1.000	0.996	0.439	1.000	0.559	0.241
2000	0.036	0.048	0.060	0.001	0.007	0.020	1.000	1.000	0.542	1.000	0.885	0.272
	m=25											
250	0.044	0.035	0.046	0.000	0.001	0.022	1.000	0.829	0.315	0.211	0.094	0.159
500	0.050	0.043	0.054	0.000	0.000	0.019	1.000	0.964	0.371	0.819	0.140	0.181
1000	0.043	0.038	0.040	0.001	0.000	0.026	1.000	0.997	0.468	0.999	0.372	0.222
2000	0.061	0.059	0.044	0.001	0.001	0.014	1.000	1.000	0.570	1.000	0.775	0.276

Table 3: **Size and power results of the CUSUM fixed-m type-I test** at different sample sizes and different d applying the local Whittle estimator. The number of replications is $1,000$.

However, when $\phi = 0.5$ we observe that the power does not monotonously increase with m. Therefore, we suggest to use a bandwidth around $m = 10$.

The CUSUM fixed-m type-II test generates power at all bandwidths even when $m = 1$. When the sample size, the memory, or the AR coefficient increase, it shows the same pattern as the type-I test. Therefore, we also suggest to use $m = 10$ for the type-II test. Comparing both tests, we see that the type-I test with larger m possesses higher power than the type-II test when d increases.

Overall, when comparing the type-I and type-II tests, the type-II demeaning increases the power if the degree of memory is low, but the additional flexibility introduced through the two-sided demeaning approach costs power if d is large so that the means are hard to identify and the resulting limit distribution is wider.

If we compare the CUSUM fixed-b with the CUSUM fixed-m tests, we find that the fixed-m tests have superior power for $\phi = 0$. However, when we have short-run dynamics in the series with an AR coefficient of $\phi = 0.5$, we observe that the fixed-b tests perform better.

	CUSUM fixed-m type II											
	size						power					
	$\phi = 0$			$\phi = 0.5$			$\phi = 0$			$\phi = 0.5$		
T/d	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4
	m=1											
250	0.014	0.019	0.031	0.021	0.050	0.048	0.145	0.092	0.083	0.135	0.127	0.074
500	0.039	0.040	0.042	0.045	0.051	0.057	0.237	0.137	0.085	0.181	0.109	0.077
1000	0.044	0.043	0.050	0.052	0.068	0.059	0.297	0.128	0.091	0.242	0.180	0.084
2000	0.072	0.059	0.052	0.064	0.063	0.060	0.422	0.162	0.093	0.288	0.147	0.096
	m=2											
250	0.046	0.042	0.046	0.029	0.026	0.038	0.830	0.359	0.134	0.320	0.163	0.098
500	0.043	0.052	0.034	0.027	0.036	0.048	0.976	0.464	0.164	0.588	0.229	0.125
1000	0.062	0.041	0.050	0.040	0.032	0.042	1.000	0.630	0.171	0.850	0.287	0.149
2000	0.062	0.046	0.040	0.061	0.035	0.042	1.000	0.777	0.191	0.979	0.431	0.169
	m=3											
250	0.066	0.054	0.048	0.017	0.023	0.043	0.976	0.507	0.193	0.429	0.154	0.100
500	0.050	0.049	0.045	0.020	0.023	0.045	0.999	0.683	0.221	0.799	0.244	0.131
1000	0.051	0.043	0.047	0.025	0.017	0.034	1.000	0.887	0.269	0.985	0.395	0.158
2000	0.055	0.043	0.048	0.029	0.031	0.026	1.000	0.969	0.315	1.000	0.621	0.183
	m=4											
250	0.041	0.070	0.046	0.006	0.020	0.030	0.994	0.592	0.188	0.468	0.144	0.106
500	0.046	0.052	0.043	0.011	0.010	0.037	1.000	0.819	0.254	0.854	0.246	0.143
1000	0.047	0.055	0.055	0.017	0.015	0.030	1.000	0.943	0.289	1.000	0.443	0.178
2000	0.047	0.067	0.059	0.019	0.029	0.041	1.000	0.994	0.359	1.000	0.681	0.181
	m=10											
250	0.054	0.052	0.042	0.002	0.003	0.026	1.000	0.779	0.287	0.396	0.111	0.145
500	0.053	0.055	0.043	0.001	0.004	0.022	1.000	0.943	0.320	0.904	0.260	0.163
1000	0.046	0.046	0.042	0.004	0.002	0.024	1.000	0.998	0.397	1.000	0.496	0.202
2000	0.054	0.057	0.043	0.003	0.007	0.021	1.000	1.000	0.482	1.000	0.803	0.274
	m=25											
250	0.042	0.043	0.044	0.000	0.002	0.025	1.000	0.811	0.280	0.163	0.061	0.159
500	0.041	0.051	0.054	0.000	0.000	0.021	1.000	0.964	0.352	0.766	0.112	0.182
1000	0.035	0.066	0.044	0.001	0.001	0.019	1.000	0.999	0.444	0.999	0.304	0.170
2000	0.042	0.059	0.054	0.001	0.000	0.011	1.000	1.000	0.547	1.000	0.741	0.234

Table 4: **Size and power results of the CUSUM fixed-m type-II test** at different sample sizes T and different long-memory parameters d applying the local Whittle estimator. The number of replications is $1,000$.

To shed light on the relative efficiency of our tests compared to alternative approaches, we conduct a local power analysis with the recommended bandwidth parameters $b = 0.1$ and $m = 10$. We compare the proposed tests with the self-normalized Wilcoxon-type test of Betken (2016) and the sup-Wald fixed-b test of Iacone, Leybourne, and Taylor (2014). The local alternative is of the form

$$\beta = kT^{d-1/2},$$

where β is the shift size and k is a finite constant, as in Iacone, Leybourne, and Taylor (2014). The results are presented as functions of k in Figure 1 for $\phi = 0$ (on the left hand side) and $\phi = 0.5$ (on the right hand side). Throughout all simulations, we use a sample size of $T = 500$.

In absence of short-run dynamics we observe that the fixed-m type-I test is the most powerful test

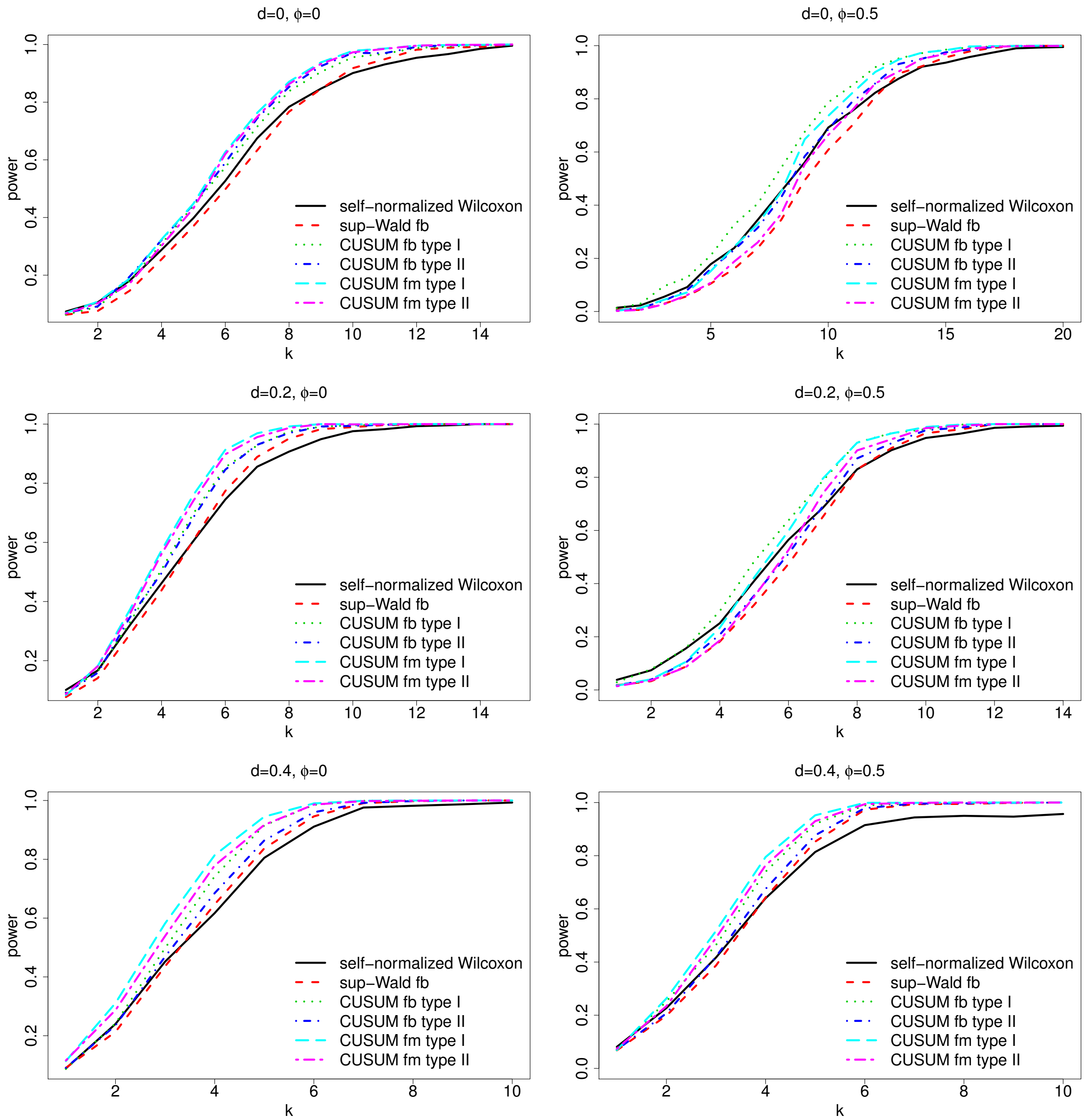

Figure 1: Local power with $\phi = 0$ (left) and $\phi = 0.5$ (right).

across all values of d and k, with a local power gain of up to 15 percentage points compared to the fixed-b sup-Wald test and up to 20 percentage points over the self-normalized Wilcoxon-type test. The fixed-m type-II test performs equally well for $d \leq 0.2$ and slightly worse for $d = 0.4$. The CUSUM fixed-b tests are worse, with the type-I test providing better power results than the type-II test. However, they still offer better results across all specifications than the best existing tests.

If $\phi = 0.5$, we observe that the CUSUM fixed-b type-I test is the most powerful test for $d \leq 0.2$. For $d = 0.4$, the CUSUM fixed-b type-I test and both CUSUM fixed-m tests perform equally well. For small d and small k the self-normalized Wilcoxon-type test also has good power properties, but overall the fixed-bandwidth CUSUM tests still provide better local power than its competitors.

Overall, we find that the self-normalized CUSUM tests outperform their competitors in terms of finite sample power, while maintaining good size properties. For practical applications we suggest to use the type-I version of the CUSUM fixed-m test with $m = 10$ that performs well for a wide range of scenarios.

4 Conclusion

While conventional CUSUM tests normalized with consistent estimates of the long–run variance are often found to be liberal in finite samples, the self-normalized versions proposed here provide good size control, while maintaining favorable power properties. The resulting test statistics therefore outperform competing approaches.

Both fixed-b and the newly proposed fixed-m approach of Hualde and Iacone (2017) are considered and we find that the fixed-m normalization based on the periodogram performs best for a wide range of scenarios.

While the type-II demeaning approach is dominated by the type-I approach if b is small or m is relatively large, practical applications with long dependence structures often require to use large b and small m. In these situations the type-II approach is favorable.

Acknowledgements

Financial support of the Deutsche Forschungsgesellschaft (DFG) is gratefully acknowledged. We are also grateful to Philipp Sibbertsen, Fabrizio Iacone, Annastiina Silvennoinen, Tristan Hirsch, Michael Stumpf, the participants of the Statistical Week 2017 in Rostock and the 26th Annual Symposium of the Society for Nonlinear Dynamics and Econometrics 2018 in Tokio for their helpful comments about earlier versions of this paper.

Appendix

Proof of Lemma 1

To prove the lemma, we start by rewriting $\widehat{\epsilon}_t^{II}(\tau)$ to show that so that

$$\begin{aligned}
\widehat{\epsilon}_t^{II}(\tau) &= y_t - I(t \leq \lfloor \tau T \rfloor)\bar{y}(\tau) - I(t > \lfloor \tau T \rfloor)\underline{y}(\tau) \\
&= \mu - I(t \leq \lfloor \tau T \rfloor)\mu - I(t > \lfloor \tau T \rfloor)\mu \\
&+ \epsilon_t - I(t \leq \lfloor \tau T \rfloor)\frac{1}{\lfloor \tau T \rfloor}\sum_{r=1}^{\lfloor \tau T \rfloor}\epsilon_r - I(t > \lfloor \tau T \rfloor)\frac{1}{\lfloor (1-\tau)T \rfloor}\sum_{l=\lfloor \tau T \rfloor+1}^{T}\epsilon_l \\
&= \epsilon_t - \frac{I(t \leq \lfloor \tau T \rfloor)}{\lfloor \tau T \rfloor}\sum_{r=1}^{\lfloor \tau T \rfloor}\epsilon_r - \frac{I(t > \lfloor \tau T \rfloor)}{\lfloor (1-\tau)T \rfloor}\sum_{l=\lfloor \tau T \rfloor+1}^{T}\epsilon_l.
\end{aligned}$$

Therefore,

$$\begin{aligned}
T^{-1/2-d}\sum_{t=1}^{\lfloor sT \rfloor}\widehat{\epsilon}_t^{II}(\tau) &= T^{-1/2-d}\sum_{t=1}^{\lfloor sT \rfloor}\epsilon_t - \frac{\sum_{t=1}^{\lfloor sT \rfloor} I(t \leq \lfloor \tau T \rfloor)}{\lfloor \tau T \rfloor}T^{-1/2-d}\sum_{r=1}^{\lfloor \tau T \rfloor}\epsilon_r \\
&- \frac{\sum_{t=1}^{\lfloor sT \rfloor} I(t > \lfloor \tau T \rfloor)}{\lfloor (1-\tau)T \rfloor}T^{-1/2-d}\sum_{s=\lfloor \tau T \rfloor+1}^{T}\epsilon_s \\
&= T^{-1/2-d}\sum_{t=1}^{\lfloor sT \rfloor}\epsilon_t - \frac{\min(s,\tau)}{\tau}T^{-1/2-d}\sum_{r=1}^{\lfloor \tau T \rfloor}\epsilon_r - \frac{\max(s-\tau,0)}{1-\tau}T^{-1/2-d}\sum_{s=\lfloor \tau T \rfloor+1}^{T}\epsilon_s
\end{aligned}$$

and by Assumption 1

$$\sigma^{-1}T^{-1/2-d}\sum_{t=1}^{\lfloor sT \rfloor}\widehat{\epsilon}_t^{II}(\tau) \Rightarrow B_d(s) - \frac{\min(s,\tau)}{\tau}B_d(\tau) - \frac{\max(s-\tau,0)}{1-\tau}(B_d(1)-B_d(\tau)) := \widetilde{B}_d^{II}(s,\tau). \quad \square$$

Proof of Theorem 1

We start by deriving the limit of the type-I estimators. Using arguments from the proof of Theorem 1 in Kiefer and Vogelsang (2005) the rescaled long-run variance estimator in (4) using the Bartlett kernel is given by

$$T^{-2d}\widehat{\sigma}_{I,b}^2 = \frac{2}{bT}\sum_{r=1}^{T-1}\left(T^{-1/2-d}\widehat{S}_{\lfloor rT \rfloor}^I\right)^2 - \frac{2}{bT}\sum_{r=1}^{T-\lfloor bT \rfloor-1}T^{-1-2d}\widehat{S}_{r+\lfloor bT \rfloor}^I\widehat{S}_r^I + o_P(1)$$

where $\widehat{S}_i^I = \sum_{t=1}^{i}\widehat{\epsilon}_t^I$. Under Assumption 1 we have

$$\begin{aligned}
T^{-1/2-d}\widehat{S}_{\lfloor \tau T \rfloor}^I = T^{-1/2-d}\sum_{t=1}^{\lfloor \tau T \rfloor}\widehat{\epsilon}_t^I = T^{-1/2-d}\left(\sum_{t=1}^{\lfloor \tau T \rfloor}\epsilon_t + \frac{\lfloor \tau T \rfloor}{T}\sum_{t=1}^{T}\epsilon_t\right) \\
\Rightarrow \sigma(B_d(\tau) - \tau B_d(1)) := \sigma\widetilde{B}_d^I(\tau).
\end{aligned}$$

Therefore, it directly follows that

$$\widehat{\sigma}^2_{I,b} \Rightarrow T^{2d}\sigma^2 \frac{2}{b}\left(\int_0^1 \widetilde{B}^I_d(r)^2 \mathrm{d}r - \int_0^{1-b} \widetilde{B}^I_d(r+b)\widetilde{B}^I_d(r)\mathrm{d}r\right) = T^{2d}\sigma^2 Q_{I,b}.$$

Now consider the fixed-m long-run variance estimator of type-I in Equation (7). Lemma 1 of Hualde and Iacone (2017) states that

$$\begin{aligned}
T^{-2d}2\pi I^I(\lambda_j) &= \left(\sum_{t=1}^{T-1}\left(-\frac{2\pi j}{T}\sin\left(2\pi j\frac{t}{T}\right) + O(\lambda_j^2)\right)T^{-1/2-d}\sum_{s=1}^{t}\widehat{\epsilon}^I_s\right)^2 \\
&+ \left(\sum_{t=1}^{T-1}\left(\frac{2\pi j}{T}\cos\left(2\pi j\frac{t}{T}\right) + O(\lambda_j^2)\right)T^{-1/2-d}\sum_{s=1}^{t}\widehat{\epsilon}^I_s\right)^2 \\
&\Rightarrow \sigma^2\left\{\left(2\pi j\int_0^1 \sin(2\pi js)\widetilde{B}^I_d(s)ds\right)^2 + \left(2\pi j\int_0^1 \cos(2\pi js)\widetilde{B}^I_d(s)ds\right)^2\right\} := \sigma^2 Q_I(j).
\end{aligned}$$

This implies

$$\widehat{\sigma}^2_{I,m} = \frac{2\pi}{m}\sum_{j=1}^{m} I^I(\lambda_j) \Rightarrow T^{2d}\frac{\sigma^2}{m}\sum_{j=1}^{m} Q_I(j) = T^{2d}\sigma^2 Q_{I,m}.$$

By analogous arguments, but replacing the fractional Brownian bridge $\widetilde{B}^I(\tau)$ of type I with its type-II version $\widetilde{B}^{II}(s,\tau)$ as defined in Lemma 1, we obtain for the type-II versions of the long-run variance estimates in Equations (5) and (8) that

$$\begin{aligned}
&\widehat{\sigma}^2_{II,b} \Rightarrow T^{2d}\sigma^2 Q_{II,b} \\
\text{and}\quad &\widehat{\sigma}^2_{II,m} \Rightarrow T^{2d}\sigma^2 Q_{II,m}.
\end{aligned}$$

From the CMT it follows that

$$\left|\frac{T^{-1/2}\widehat{S}^I_{\lfloor \tau T\rfloor}}{\widehat{\sigma}_{I,\delta}}\right| \Rightarrow \left|\frac{\sigma T^{1/2+d}\widetilde{B}^I_d(\tau)}{T^d\sigma\sqrt{Q_{I,\delta}}}\right| = \left|\frac{\widetilde{B}^I_d(\tau)}{\sqrt{Q_{I,\delta}}}\right| \quad \text{and} \quad \left|\frac{T^{-1/2}\widehat{S}^I_{\lfloor \tau T\rfloor}}{\widehat{\sigma}_{II,\delta}}\right| \Rightarrow \left|\frac{\widetilde{B}^I_d(\tau)}{\sqrt{Q_{II,\delta}(s)}}\right|.$$

Since the supremum is a continuous function of its arguments it follows that

$$Z^I_\delta \Rightarrow \sup_{\tau\in[\tau_1,\tau_2]}\left|\frac{\widetilde{B}^I_d(\tau)}{\sqrt{Q_{I,\delta}}}\right| \quad \text{and} \quad Z^{II}_\delta \Rightarrow \sup_{\tau\in[\tau_1,\tau_2]}\left|\frac{\widetilde{B}^I_d(\tau)}{\sqrt{Q_{I,\delta}(\tau)}}\right|. \qquad \square$$

b	level/d	-0.49	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4	0.49
	10%	0.796	0.847	0.905	0.982	1.085	1.200	1.315	1.433	1.563	1.662	1.758
0.05	5%	0.846	0.906	0.978	1.071	1.174	1.302	1.421	1.545	1.674	1.769	1.850
	1%	0.956	1.030	1.116	1.215	1.354	1.504	1.608	1.729	1.851	1.921	1.989
	10%	1.008	1.041	1.072	1.108	1.146	1.188	1.238	1.275	1.314	1.356	1.391
0.1	5%	1.071	1.112	1.143	1.185	1.229	1.271	1.316	1.349	1.391	1.423	1.450
	1%	1.192	1.234	1.275	1.320	1.351	1.406	1.444	1.464	1.494	1.514	1.525
	10%	1.269	1.259	1.243	1.243	1.217	1.204	1.188	1.172	1.161	1.152	1.147
0.2	5%	1.332	1.324	1.307	1.305	1.274	1.258	1.241	1.224	1.204	1.192	1.184
	1%	1.460	1.440	1.427	1.426	1.392	1.359	1.332	1.318	1.280	1.266	1.248
	10%	1.447	1.427	1.387	1.343	1.300	1.254	1.213	1.173	1.137	1.115	1.092
0.3	5%	1.513	1.498	1.462	1.409	1.373	1.320	1.272	1.231	1.192	1.163	1.134
	1%	1.669	1.643	1.590	1.547	1.502	1.457	1.403	1.359	1.313	1.273	1.237
	10%	1.607	1.572	1.521	1.461	1.405	1.352	1.299	1.244	1.200	1.160	1.131
0.4	5%	1.685	1.646	1.602	1.532	1.487	1.426	1.374	1.320	1.263	1.223	1.190
	1%	1.851	1.811	1.766	1.700	1.654	1.586	1.538	1.473	1.398	1.372	1.338
	10%	1.756	1.713	1.661	1.600	1.535	1.465	1.399	1.338	1.295	1.245	1.213
0.5	5%	1.844	1.795	1.750	1.686	1.621	1.550	1.488	1.420	1.379	1.321	1.300
	1%	2.016	1.960	1.933	1.872	1.793	1.720	1.670	1.597	1.539	1.476	1.463
	10%	1.888	1.844	1.797	1.710	1.644	1.579	1.495	1.441	1.388	1.327	1.292
0.6	5%	1.984	1.936	1.897	1.809	1.743	1.680	1.592	1.529	1.488	1.413	1.388
	1%	2.185	2.141	2.089	2.003	1.924	1.860	1.780	1.704	1.665	1.588	1.561
	10%	2.020	1.969	1.907	1.829	1.754	1.667	1.594	1.524	1.463	1.413	1.371
0.7	5%	2.120	2.073	2.007	1.936	1.851	1.772	1.694	1.621	1.562	1.504	1.454
	1%	2.327	2.281	2.197	2.142	2.042	1.990	1.889	1.805	1.748	1.684	1.634
	10%	2.150	2.091	2.027	1.944	1.852	1.767	1.686	1.614	1.541	1.492	1.449
0.8	5%	2.260	2.201	2.130	2.050	1.951	1.861	1.794	1.711	1.632	1.578	1.540
	1%	2.475	2.435	2.352	2.272	2.143	2.063	1.994	1.904	1.831	1.752	1.739
	10%	2.267	2.215	2.142	2.047	1.957	1.862	1.776	1.698	1.638	1.571	1.523
0.9	5%	2.380	2.326	2.252	2.159	2.063	1.977	1.878	1.804	1.727	1.660	1.617
	1%	2.611	2.557	2.463	2.362	2.286	2.209	2.099	1.985	1.921	1.842	1.798
	10%	2.375	2.312	2.248	2.160	2.065	1.965	1.869	1.796	1.721	1.654	1.608
1	5%	2.492	2.430	2.359	2.274	2.183	2.071	1.975	1.903	1.833	1.756	1.709
	1%	2.749	2.674	2.603	2.521	2.405	2.293	2.186	2.128	2.026	1.953	1.898

Table 5: **Asymptotic critical values for the CUSUM fixed-b type-I test** in (9), using the Bartlett kernel and $[\tau_1, \tau_2] = [0.15, 0.85]$. These are obtained from 10,000 Monte Carlo replications with $T = 1,000$.

b	level/d	-0.49	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4	0.49
	10%	0.807	0.857	0.941	1.053	1.192	1.370	1.577	1.862	2.156	2.513	2.907
0.05	5%	0.865	0.923	1.031	1.161	1.326	1.536	1.803	2.139	2.476	2.898	3.350
	1%	0.983	1.083	1.204	1.369	1.606	1.876	2.242	2.714	3.247	3.675	4.347
	10%	1.018	1.074	1.155	1.253	1.375	1.547	1.712	1.932	2.185	2.518	2.754
0.1	5%	1.090	1.161	1.277	1.383	1.551	1.750	1.983	2.247	2.560	2.949	3.327
	1%	1.239	1.333	1.492	1.692	1.919	2.184	2.580	2.910	3.346	3.896	4.472
	10%	1.228	1.305	1.423	1.554	1.700	1.889	2.087	2.322	2.620	2.900	3.146
0.2	5%	1.320	1.419	1.577	1.732	1.950	2.184	2.456	2.744	3.121	3.464	3.833
	1%	1.517	1.679	1.864	2.121	2.489	2.839	3.348	3.601	4.283	4.899	5.367
	10%	1.348	1.449	1.622	1.764	1.978	2.200	2.484	2.805	3.125	3.487	3.735
0.3	5%	1.455	1.584	1.802	1.985	2.295	2.599	2.963	3.312	3.764	4.259	4.548
	1%	1.673	1.895	2.172	2.485	2.937	3.444	3.990	4.474	5.388	5.894	6.365
	10%	1.444	1.565	1.730	1.976	2.244	2.536	2.858	3.180	3.581	4.015	4.412
0.4	5%	1.574	1.730	1.951	2.241	2.618	2.989	3.372	3.843	4.365	4.865	5.345
	1%	1.852	2.085	2.397	2.797	3.364	3.937	4.598	5.269	5.987	6.840	7.333
	10%	1.556	1.697	1.877	2.156	2.438	2.852	3.201	3.575	4.125	4.549	4.868
0.5	5%	1.700	1.888	2.112	2.479	2.818	3.377	3.746	4.326	5.016	5.473	5.842
	1%	2.026	2.282	2.652	3.177	3.724	4.481	5.094	5.903	6.944	7.618	8.205
	10%	1.659	1.817	2.034	2.323	2.659	3.054	3.472	3.924	4.424	4.940	5.364
0.6	5%	1.841	2.016	2.305	2.676	3.121	3.624	4.113	4.679	5.354	5.878	6.454
	1%	2.225	2.478	2.831	3.424	4.020	4.892	5.558	6.326	7.433	8.277	9.021
	10%	1.779	1.928	2.175	2.480	2.840	3.230	3.701	4.224	4.781	5.270	5.731
0.7	5%	1.990	2.165	2.467	2.889	3.304	3.846	4.416	5.085	5.688	6.398	6.866
	1%	2.376	2.675	3.117	3.731	4.287	5.008	6.014	6.910	7.708	8.922	9.534
	10%	1.882	2.043	2.316	2.628	2.997	3.427	3.981	4.427	4.954	5.609	5.940
0.8	5%	2.084	2.308	2.638	3.061	3.476	4.064	4.784	5.385	6.008	6.676	7.413
	1%	2.496	2.886	3.348	3.897	4.494	5.374	6.398	7.234	8.456	9.108	10.302
	10%	2.004	2.211	2.429	2.751	3.175	3.605	4.102	4.631	5.252	5.800	6.286
0.9	5%	2.258	2.503	2.783	3.202	3.691	4.281	4.906	5.567	6.348	6.985	7.574
	1%	2.723	3.051	3.515	4.072	4.798	5.692	6.877	7.479	8.818	9.630	10.363
	10%	2.134	2.323	2.610	2.964	3.323	3.786	4.346	4.893	5.443	6.059	6.577
1	5%	2.363	2.611	2.974	3.407	3.888	4.443	5.181	5.853	6.524	7.308	7.956
	1%	2.849	3.189	3.761	4.334	5.118	5.944	7.109	8.140	9.041	10.014	11.333

Table 6: **Asymptotic critical values for the CUSUM fixed-b type-II test** in (10), using the Bartlett kernel and $[\tau_1, \tau_2] = [0.15, 0.85]$. These are obtained from 10,000 Monte Carlo replications with $T = 1,000$.

m	level/d	-0.49	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4	0.49
	10%	4.386	3.725	3.097	2.655	2.294	2.024	1.781	1.529	1.437	1.292	1.227
1	5%	6.209	5.157	4.365	3.723	3.121	2.780	2.453	2.038	1.954	1.675	1.600
	1%	14.475	12.329	9.994	8.134	7.119	6.479	5.397	3.980	4.009	3.196	3.456
	10%	2.457	2.145	1.864	1.623	1.465	1.339	1.235	1.159	1.120	1.082	1.070
2	5%	2.956	2.570	2.208	1.913	1.698	1.521	1.389	1.297	1.228	1.177	1.162
	1%	4.365	3.828	3.260	2.815	2.362	2.102	1.829	1.738	1.609	1.479	1.428
	10%	1.919	1.727	1.551	1.393	1.305	1.236	1.186	1.160	1.151	1.150	1.150
3	5%	2.196	1.957	1.745	1.563	1.452	1.352	1.288	1.253	1.230	1.219	1.208
	1%	2.992	2.633	2.212	1.983	1.790	1.649	1.535	1.437	1.406	1.361	1.326
	10%	1.639	1.526	1.383	1.298	1.247	1.214	1.195	1.195	1.205	1.228	1.248
4	5%	1.846	1.705	1.543	1.419	1.366	1.318	1.290	1.281	1.285	1.295	1.311
	1%	2.356	2.118	1.905	1.717	1.616	1.525	1.491	1.432	1.426	1.412	1.408
	10%	1.066	1.040	1.045	1.069	1.118	1.188	1.290	1.394	1.496	1.605	1.702
10	5%	1.163	1.141	1.146	1.172	1.222	1.304	1.411	1.517	1.612	1.720	1.811
	1%	1.335	1.339	1.363	1.379	1.438	1.519	1.610	1.721	1.802	1.888	1.964
	10%	0.715	0.755	0.817	0.907	1.028	1.187	1.394	1.641	1.907	2.185	2.415
25	5%	0.770	0.818	0.887	0.988	1.138	1.318	1.537	1.820	2.100	2.384	2.588
	1%	0.884	0.942	1.050	1.173	1.340	1.557	1.791	2.107	2.415	2.668	2.854
	10%	0.547	0.598	0.683	0.803	0.965	1.205	1.511	1.872	2.283	2.801	3.175
50	5%	0.586	0.644	0.745	0.882	1.070	1.335	1.674	2.096	2.528	3.072	3.447
	1%	0.669	0.736	0.869	1.039	1.266	1.587	1.980	2.447	2.982	3.505	3.828
	10%	0.423	0.482	0.577	0.713	0.911	1.199	1.609	2.131	2.789	3.549	4.239
100	5%	0.457	0.521	0.623	0.784	1.007	1.334	1.782	2.385	3.115	3.883	4.615
	1%	0.525	0.597	0.728	0.913	1.201	1.585	2.160	2.823	3.668	4.478	5.182
	10%	0.372	0.430	0.528	0.664	0.879	1.190	1.638	2.253	3.086	4.070	5.003
150	5%	0.399	0.462	0.574	0.728	0.968	1.329	1.828	2.522	3.453	4.542	5.476
	1%	0.450	0.530	0.665	0.849	1.138	1.589	2.231	3.029	4.140	5.291	6.130
	10%	0.335	0.394	0.484	0.633	0.845	1.177	1.670	2.367	3.294	4.471	5.566
200	5%	0.359	0.424	0.526	0.695	0.939	1.311	1.864	2.658	3.694	4.967	6.121
	1%	0.410	0.479	0.617	0.812	1.100	1.547	2.215	3.249	4.436	5.844	6.992

Table 7: **Asymptotic critical values for the CUSUM fixed-m type-I test** in (9), using $[\tau_1, \tau_2] = [0.15, 0.85]$. These are obtained from 10,000 Monte Carlo replications with $T = 1,000$.

m	level/d	-0.49	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4	0.49
	10%	34.984	37.289	36.392	36.630	36.914	31.743	27.949	22.665	21.577	18.019	16.280
1	5%	53.358	55.488	57.608	59.548	61.709	58.527	55.304	50.254	44.972	39.246	31.455
	1%	130.187	136.175	139.679	154.724	155.474	147.528	160.570	154.650	176.660	160.520	135.758
	10%	2.531	2.551	2.632	2.706	2.710	2.845	2.923	2.942	3.289	3.387	3.534
2	5%	3.336	3.457	3.446	3.652	3.679	3.739	3.820	3.910	4.399	4.623	4.669
	1%	6.404	6.908	6.565	6.786	6.802	7.230	7.120	6.833	7.529	8.093	8.421
	10%	1.770	1.800	1.807	1.833	1.883	1.965	2.073	2.175	2.383	2.530	2.757
3	5%	2.030	2.103	2.144	2.226	2.266	2.426	2.507	2.682	2.967	3.152	3.476
	1%	2.833	3.035	3.058	3.184	3.456	3.733	3.726	4.122	4.565	4.925	5.249
	10%	1.559	1.554	1.593	1.604	1.650	1.718	1.851	1.963	2.148	2.405	2.612
4	5%	1.736	1.745	1.816	1.844	1.917	2.012	2.221	2.339	2.590	2.886	3.177
	1%	2.167	2.234	2.328	2.477	2.654	2.800	3.210	3.261	3.628	4.131	4.542
	10%	1.077	1.086	1.107	1.162	1.243	1.388	1.555	1.749	2.053	2.400	2.710
10	5%	1.182	1.188	1.220	1.288	1.391	1.587	1.776	2.026	2.359	2.784	3.138
	1%	1.430	1.441	1.469	1.578	1.723	2.009	2.234	2.585	2.966	3.548	4.075
	10%	0.724	0.765	0.833	0.938	1.075	1.290	1.538	1.866	2.326	2.839	3.433
25	5%	0.786	0.829	0.906	1.033	1.193	1.442	1.737	2.134	2.653	3.267	3.942
	1%	0.904	0.964	1.052	1.220	1.433	1.746	2.119	2.673	3.364	4.150	4.979
	10%	0.547	0.601	0.687	0.813	0.986	1.234	1.582	2.046	2.654	3.396	4.230
50	5%	0.589	0.649	0.745	0.892	1.091	1.376	1.789	2.300	3.018	3.892	4.892
	1%	0.684	0.744	0.882	1.055	1.312	1.663	2.194	2.861	3.783	4.913	6.126
	10%	0.425	0.484	0.580	0.715	0.923	1.220	1.620	2.239	3.064	4.175	5.433
100	5%	0.455	0.525	0.630	0.790	1.015	1.365	1.802	2.540	3.478	4.774	6.192
	1%	0.517	0.600	0.730	0.934	1.196	1.639	2.211	3.173	4.347	5.984	7.722
	10%	0.371	0.426	0.531	0.672	0.886	1.201	1.670	2.360	3.320	4.738	6.171
150	5%	0.400	0.460	0.581	0.731	0.968	1.328	1.875	2.692	3.800	5.406	7.113
	1%	0.454	0.521	0.674	0.852	1.163	1.604	2.325	3.271	4.755	6.722	8.898
	10%	0.339	0.396	0.488	0.640	0.848	1.180	1.691	2.437	3.560	5.077	6.874
200	5%	0.363	0.425	0.531	0.694	0.938	1.315	1.890	2.744	4.013	5.770	7.880
	1%	0.411	0.491	0.612	0.802	1.127	1.576	2.328	3.378	4.978	7.157	9.874

Table 8: **Asymptotic critical values for the CUSUM fixed-m type-II test** in (10), using $[\tau_1, \tau_2] = [0.15, 0.85]$. These are obtained from 10,000 Monte Carlo replications with $T = 1,000$.

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