# Convergence of Adaptive Learning and the Concept of Expectational Stability in Linear Rational Expectations Models with Multiple Equilibria

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## Abstract

This paper analyzes the relationship between the expectational stability of rational expectations equilibria and the possible convergence of adaptive learning processes. Both concepts are used as selection citeria in the case of multiple rational expectations equilibria. Results obtained using recursive least squares lead to the conjecture that a correspondence between these both selection criteria exists. This paper demonstrates on the basis of a simple linear model and an alternative learning procedure that such a conjecture would be incorrect: there exist learning procedures that may converge to rational expectations equilibria which are not expectationally stable.

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# 1 Introduction

Dynamic economic models, where agents have to form expectations regarding the future, are usually closed by assuming that agents form rational expectations. Although this concept is formally elegant, it is not without problems. One of these problems is due to the fact that economic models may exhibit multiple rational expectations equilibria. In such a case it is impossible to select a specific equilibrium without imposing additional restrictions. This is especially unpleasant if the equilibria differ with respect to their comparative-static properties. Thus, economic literature (e.g. McCallum (1983), Evans and Honkapohja (1992)) discusses some selection criteria that should help to select a unique rational expectations equilibrium in such cases.

One of these selection criteria requires that the respective equilibria are possible results of adaptive learning processes on the side of the agents in the model. Under the usual assumption of bounded rational learning, agents use parametrically specified — but possibly misspecified — auxiliary models of the economy and form their expectations on the basis of these models. Learning in this context then means that agents recursively estimate the parameters of their auxiliary models. A central result of this line of research is that the possible convergence of learning processes is connected with the expectational stability (E-stability) of the respective equilibria. The concept of E-stability is another criterion proposed to select among multiple rational expectations equilibria. It traces back to Lucas (1978) and DeCanio (1979), while the extension of this concept to the form used in the context of learning is attributed to Evans (1989). Assuming that agents learn using recursive least squares, Marcet and Sargent (1988,1989) and Evans and Honkapohja (1994,1995) show that only E-stable rational expectations equilibria are possible outcomes of such learning processes.

This result suggests that the convergence analysis of learning processes and the selection of equilibria may be simultaneously achieved simply by checking the E-stability of the respective equilibria. Such a result would be very comfortable, because it is usually much simpler to check the E-stability of an equilibrium than the possible convergence of learning processes.

As is shown in this paper, however, such a conclusion cannot be drawn in general, because the results obtained with least squares learning do not generalize to other adaptive learning procedures. On the basis of a simple linear model exhibiting multiple rational expectations equilibria and using an alternative learning procedure, it is shown that the possible convergence of this learning procedure to rational expectations equilibria is independent of the E-stability of these equilibria.

The model considered here also demonstrates that the selection of equilibria using a criterion that requires the convergence of learning processes may not give definite answers:[1] it may not be possible to select a unique equilibrium and, moreover, it is quite possible that different learning procedures select different equilibria.

# 2 A linear model

In what follows, I consider a model which is equivalent to that of Evans and Honkapohja (1995) to describe the concept of E-stability and its connection to the possible convergence of adaptive learning procedures.[2] It is assumed that the value of the endogenous variable $y$ in period $t$ depends on its expected values for the periods $t$ and $t+1$ as well as an exogenous, serial uncorrelated disturbance $x_t$:

$$y_t = \beta_0 + \beta_1 y_t^e + \beta_2 y_{t+1}^e + x_t \tag{1}$$

With respect to the exogenous variable it is assumed that $\mathrm{E}[x_t] = 0$ and $\mathrm{E}[x_t]^2 = \mathrm{Var}[x_t] = M_x$ for all $t$. The expectations appearing in (1) have to be formed before $y_t$ is known and the relevant information set is given as $\Omega_t = \{y_{t-1}, y_{t-2}, \ldots, x_{t-1}, x_{t-2}, \ldots\}$. This implies that the exogenous disturbances $x_t$ are observable for all $t$ — although they are observable only after the relevant expectations in $t$ have been formed.

As is shown by Evans (1985), the model exhibits multiple rational expectations equilibria that may be computed from the general ARMA(1,1) solution

$$y_t = a_0 + a_1 y_{t-1} + a_2 x_{t-1} + x_t$$

imposing parameter restrictions.[3] The two solution sets are given by:

$$y_t = \frac{\beta_0}{1-\beta_1-\beta_2} + x_t \tag{2-a}$$

$$y_t = -\frac{\beta_0}{\beta_2} + \frac{1-\beta_1}{\beta_2} y_{t-1} + \phi x_{t-1} + x_t\,, \quad \phi \in \mathbb{R} \tag{2-b}$$

Note that the white noise solution (2-a) is unique, whereas the ARMA(1,1) solution (2-b) represents a continuum of solutions for (1), because $\phi$ can take any value.

[1] A similar point is made by Duffy (1994). He shows on the basis of an OLG-model that this criterion may not be selective.

[2] Evans and Honkapohja (1995) give a comprehensive analysis of this model with respect to E-stability and the convergence of recursive least squares learning.

[3] In what follows, other solutions that can be obtained by taking into account sunspot variables are ignored. See Evans and Honkapohja (1992,1995).

# 3 Expectational Stability

The starting point for the concept of E-stability is an auxiliary model which describes the perceived connection between the endogenous and the exogenous variables. In what follows, this auxiliary model takes the form of an ARMA(1,1) model:

$$y_t = \alpha + \psi y_{t-1} + \phi x_{t-1} + x_t \tag{3}$$

Given this auxiliary model, expectations regarding the future values of the endogenous variable are given as:

$$\begin{aligned} y_t^e &= \mathrm{E}[y_t \,|\, y_{t-1}, x_{t-1}] = \alpha + \psi y_{t-1} + \phi x_{t-1} \\ y_{t+1}^e &= \mathrm{E}[y_{t+1} \,|\, y_{t-1}, x_{t-1}] = \alpha + \psi \,\mathrm{E}[y_t \,|\, y_{t-1}, x_{t-1}] \\ &= (1+\psi)\,\alpha + \psi^2 y_{t-1} + \psi\phi x_{t-1} \end{aligned}$$

Substituting these equations into the model (1) yields the following actual process for the endogenous variable $y_t$:

$$\begin{aligned} y_t &= \beta_0 + \alpha\,[\beta_1 + \beta_2(1+\psi)] + \left[\beta_1\psi + \beta_2\psi^2\right] y_{t-1} + [\beta_1\phi + \beta_2\psi\phi]\, x_{t-1} + x_t \\ &= T_\alpha(\theta) + T_\psi(\theta)\, y_{t-1} + T_\phi(\theta)\, x_{t-1} + x_t \end{aligned}$$

Here $\theta' = (\alpha, \psi, \phi)$ denotes the vector of parameters of the auxiliary model and the operator $T(\theta)' = (T_\alpha(\theta), T_\psi(\theta), T_\phi(\theta))$ maps the parameter vector $\theta$ of the auxiliary model (3) to the parameters of the resulting actual process for $y_t$. The operator $T(\theta)$ is thus given as:

$$T(\theta) = T\begin{pmatrix}\alpha\\ \psi\\ \phi\end{pmatrix} = \begin{pmatrix}T_\alpha(\theta)\\ T_\psi(\theta)\\ T_\phi(\theta)\end{pmatrix} = \begin{pmatrix}\beta_0 + \alpha\,[\beta_1 + \beta_2(1+\psi)]\\ \beta_1\psi + \beta_2\psi^2\\ \beta_1\phi + \beta_2\phi\psi\end{pmatrix}, \tag{4}$$

and from (4) we may compute the following stationary points of $T(\theta)$:

$$\theta_I'^{*} = (\beta_0\,(1-\beta_1-\beta_2)^{-1}, 0, 0) \tag{5-a}$$

$$\Theta_{II}^{*} = \left\{\theta \,|\, \theta' = (-\beta_0\,\beta_2^{-1}, (1-\beta_1)\,\beta_2^{-1}, \phi), \quad \phi \in \mathbb{R}\right\} \tag{5-b}$$

It is apparent that the stationary points (5-a) and (5-b) imply that the auxiliary model (3) coincides with the rational expectations equilibria of (1).

Following Evans (1989), a rational expectations equilibrium is E-stable if this equilibrium is (locally) asymptotically stable with respect to the differential equation

$$\frac{d\theta}{d\tau} = \dot{\theta} = T(\theta) - \theta \tag{6}$$

based on the operator $T(\theta)$. This in turn requires that the stationary points (5-a) and (5-b) associated with this respective equilibrium are (locally) stable stationary points of the differential equation (6).

Whether or not a specific rational expectations solution is E-stable may depend upon the underlying auxiliary model used by the agents. The criterion of E-stability thus distinguishes between the case of correctly parametrized and overparametrized auxiliary models. A solution is said to be weakly E-stable if it is stable given an auxiliary model which is correctly parametrized, it is strongly E-stable if it is also stable given an overparametrized auxiliary model. Thus, given the auxiliary model (3) it is clear that we have to check whether or not the solution (2-a) is strongly E-stable and whether or not (2-b) is weakly E-stable.

The Jacobian matrix of the differential equation (6) is given as:

$$J(\theta) = \nabla_\theta T(\theta) - \mathrm{I}_3 = \begin{pmatrix} \beta_1+\beta_2(1+\psi)-1 & \alpha\beta_2 & 0 \\ 0 & \beta_1+2\beta_2\psi-1 & 0 \\ 0 & \beta_2\phi & \beta_1+\beta_2\psi-1 \end{pmatrix}, \qquad (7)$$

Therefore the resulting eigenvalues $\mu_1, \mu_2, \mu_3$ of $J(\theta)$ are:

$$\begin{aligned} \mu_1 &= \beta_1 + \beta_2\,(1+\psi) - 1\,, \\ \mu_2 &= \beta_1 + 2\,\beta_2\,\psi - 1\,, \\ \mu_3 &= \beta_1 + \beta_2\,\psi - 1 \end{aligned}$$

Regarding the stationary point $\theta_I^*$ from (5-a), it follows that $\mu_1 = \beta_1 + \beta_2 - 1$ and $\mu_2 = \mu_3 = \beta_1 - 1$. Thus, the rational expectations equilibrium (2-a) is strongly E-stable if:

$$\beta_1 < 1\,, \quad \beta_1 + \beta_2 < 1$$

Note that for the white noise solution (2-a) to be weakly E-stable it is sufficient that $\beta_1 + \beta_2 < 1$.

With respect to stationary points $\theta^* \in \Theta_{II}^*$ from (5-b), we get $\mu_1 = \beta_2$, $\mu_2 = 1 - \beta_1$ as well as $\mu_3 = 0$. The corresponding rational expectations equilibrium (2-b) is weakly E-stable if:[4]

$$\beta_1 > 1\,, \quad \beta_2 < 0$$

As shown by Evans and Honkapohja (1994), this rational expectations equilibrium will never be strongly E-stable.

[4] Because one eigenvalue of $J(\theta^*)$ equals zero, the stability analysis is more difficult in this case. But as shown by Evans and Honkapohja (1992), the above stated conditions ensure weak E-stability for all $\theta^* \in \Theta_{II}^*$.

**Figure 1:** *Expectational stability of (2-a) and (2-b) dependent on* $\beta_1$ *and* $\beta_2$

As figure 1 makes clear, these stability conditions separate the parameter space of the model, such that given the auxiliary model (3) either only one of the solutions (2-a) or (2-b) is E-stable or none of these solutions is E-stable. Thus, dependent on the values of the parameters $\beta_1$ and $\beta_2$ we can distinguish three regions in figure 1. In region ① the only E-stable equilibrium is (2-a), whereas in region ② only the ARMA solutions (2-b) are E-stable. Finally, in region ③ none of the two equilibria is E-stable. Two additional points have to be noted here: first, while it is true that the equilibrium (2-a) is not strongly E-stable in region ②, there still exist parameters $\beta_1$ and $\beta_2$ satisfying $\beta_1 + \beta_2 < 1$ sucht that (2-a) is at least weakly E-stable. Second, the ARMA solution (2-b) will be stationary only if $|(1-\beta_1)\beta_2^{-1}| < 1$.

# 4 Analysis of learning processes

## 4.1 Learning via recursive parameter estimation

Now assume that agents in the model use auxiliary models of the form (3) to form the expectations relevant in (1). Given a vector $\theta_t$ of parameters in period $t$, this vector is modified by the agents if new observations of the exogenous and endogenous variables become available.

With $z_t' = (1, y_{t-1}, x_{t-1})$ the forecast of the endogenous variable in period $t$ (after $x_t$ has been oberved) computed in the auxiliary model and denoted by $\hat{y}_t$, is given as $\hat{y}_t = z_t' \theta_t + x_t$. On the other hand, expectations formed on the basis of the auxiliary model imply that the actual value for the endogenous variable is $y_t = z_t' T(\theta_t) + x_t$. Regarding the vector $z_t$, this results in the following stochastic process:

$$z_{t+1} = \begin{pmatrix} 1 \\ y_t \\ x_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ T_\alpha(\theta_t) & T_\psi(\theta_t) & T_\phi(\theta_t) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} x_t$$

$$= A(\theta_t)\, z_t + b\, x_t \tag{8}$$

Note that given $\theta_t$, the stochastic process (8) is stationary only if $|T_\psi(\theta_t)| < 1$. The stationarity of (8) is an important assumption for the convergence analysis of learning processes that follows. Note that this condition is not satisfied for any nonstationary ARMA solution (2-b).

Regarding the estimation procedures or learning algorithms agents use to adapt their parameter vector $\theta_t$ in the light of past experience, two alternative procedures are considered:[5]

(1) *Stochastic Gradient Procedure / SG Algorithm:*
The SG algorithm modifies the parameter vector based on the gradient of the forecast $\hat{y}_t = z_t'\,\theta_t + x_t$ with respect to $\theta_t$ and the forecast error $y_t - \hat{y}_t = z_t'[T(\theta_t) - \theta_t]$. We get:

$$\theta_{t+1} = \theta_t + \gamma_{t+1}\, z_t\, \left\{ z_t'\, [T(\theta_t) - \theta_t] \right\} \tag{9}$$

where $\gamma_t = t^{-\kappa}$ with $0 < \kappa \leq 1$ is a time dependent and declining learning rate. Such algorithms are quite common, especially for recursive estimation of nonlinear models like neural networks (see Sargent (1993), Kuan and White (1994)).

(2) *Recursive Least Squares / LS Algorithm:*
The main difference between the SG algorithm and recursive least squares is that the latter algorithm takes the inverse of the estimated moment matrix of the explanatory variables to determine the stepsize of the parameter modification. The LS algorithm is thus given as:[6]

$$\theta_{t+1} = \theta_t + \left(\frac{1}{t+1}\right) R_{t+1}^{-1} z_t\, \left\{ z_t'\, [T(\theta_t) - \theta_t] \right\} \tag{10-a}$$

$$R_{t+1} = R_t + \left(\frac{1}{t+1}\right) \left[ z_t\, z_t' - R_t \right] \tag{10-b}$$

If certain assumptions are satisfied, we can follow Ljung (1977) and the asymptotic properties of learning algorithms such as (9) and (10) can be described by means of an associated

[5] For a discussion of these and other algorithms see Ljung and Söderström (1983).

[6] In accordance with the SG algorithm, this algorithm may be modified by using the more general learning rate $\gamma_{t+1} = (t+1)^{-\kappa}$ with $0 < \kappa \leq 1$ instead of $(1/(t+1))$. This would result in a weighted least squares estimator.

deterministic differential equation. Since comprehensive descriptions of this approach and the required assumptions are already available in economic literature (Marcet and Sargent (1989), Woodford (1990)) and because most of these assumptions are already satisfied by the model considered here (see Evans and Honkapohja (1995)), only those assumptions that are not yet fulfilled are mentioned.

First, it is assumed that the exogenous variables $x$ are for all $t$ realizations of bounded random variables. Second, the parameter space for $\theta$ is restricted to a set $D_\theta \subseteq \mathbb{R}^3$, such that for all $\theta \in D_\theta$ the matrix $A(\theta)$ has the property that the resulting stochastic process for $z_t$ from (8) is stationary. As mentioned above, this condition is satisfied, whenever $|T_\psi(\theta)| < 1$. These two assumptions imply (see Woodford (1990)) that a bounded set $D_z \subset \mathbb{R}^3$ exists such that for all $z_0 \in D_z$ we have $z_t \in D_z$ for all $t$ with probability one.

Given a fixed $\theta \in D_\theta$, let $M_z(\theta)$ denote the $(3 \times 3)$ matrix of moments $\mathrm{E}[z_t\, z_t']$. From the above made assumptions it then follows that $M_z(\theta)$ is bounded and given as:[7]

$$M_z(\theta) = \mathrm{E}[z_t\, z_t'] = \begin{pmatrix} 1 & \frac{T_\alpha(\theta)}{1-T_\psi(\theta)} & 0 \\ \frac{T_\alpha(\theta)}{1-T_\psi(\theta)} & \left(\frac{T_\alpha(\theta)}{1-T_\psi(\theta)}\right)^2 + \frac{1+T_\phi(\theta)^2+2T_\psi(\theta)T_\phi(\theta)}{1-T_\psi(\theta)^2} M_x & M_x \\ 0 & M_x & M_x \end{pmatrix} \tag{11}$$

The differential equation associated with the SG algorithm (9) can now be formulated as:

$$\frac{d\theta}{d\tau} = \dot{\theta} = \mathrm{E}[z_t\, z_t']\,[T(\theta) - \theta] = M_z(\theta)\;[T(\theta) - \theta] \tag{12}$$

Regarding the LS algorithm, the resulting associated differential equations are:

$$\frac{d\theta}{d\tau} = \dot{\theta} = R^{-1}\, M_z(\theta)\;[T(\theta) - \theta] \tag{13-a}$$

$$\frac{dR}{d\tau} = \dot{R} = M_z(\theta) - R \tag{13-b}$$

The essential result of Ljung (1977) states that the parameter vector $\theta_t$ resulting from learning algorithms such as (9) and (10) will — if at all — only converge to stationary points of these associated differential equations. Moreover, the probability for such a convergence to occur is positive only if the respective stationary points are (locally) stable. Convergence

[7] Because $\mathrm{E}[x_t] = 0$ we get $\mathrm{E}[y_t\, x_t] = M_x$, $\mathrm{E}[y_t\, x_{t-1}] = (T_\psi + T_\phi)\, M_x$ and $\mathrm{E}[y_t] = T_\alpha/(1-T_\psi)$. From the two equations:

$$\mathrm{E}[y_t\, y_{t-1}] = T_\alpha\, \mathrm{E}[y_t] + T_\psi\, \mathrm{E}[y_{t-1}^2] + T_\phi\, M_x\,,$$
$$\mathrm{E}[y_t^2] = T_\alpha\, \mathrm{E}[y_t] + T_\psi\, \mathrm{E}[y_t\, y_{t-1}] + T_\phi\, \mathrm{E}[y_t\, x_{t-1}] + M_x\,,$$

it then follows that $(1-T_\psi^2)\,\mathrm{E}[y_t^2] = T_\alpha\,(1+T_\psi)\,\mathrm{E}[y_t] + [1 + T_\phi^2 + 2\,T_\phi T_\psi]\, M_x$.

with probability one (almost sure convergence) can be guaranteed only if additional restrictions are employed: $\theta_t$ will converge almost surely to a stationary point of the associated differential equation if the learning algorithm is equipped with a so called projection facility. This projection guarantees that the estimated parameter vector will almost surely stay infinitely often in the domain of attraction of that stationary point. Given a stable stationary point of the associated differential equation, it is always possible to find a nontrivial set (i.e. a set containing not only the stationary point itself) having this property, such that one can always find a nontrivial projection facility ensuring almost sure convergence (see Marcet and Sargent (1989)).

However, with respect to the model considered here, a remark is necessary, because the convergence results that are derived with the help of associated differential equations are not valid if the respective stationary points belong to an unbounded continuum. As will be shown below, exactly this is the case here. But irrespective of this, at least the following statements about the algorithms and their associated differential equations are possible: the algorithms considered here will not converge to parameter vectors that are not stationary points of the associated differential equation and they will not converge to stationary points that are unstable.

Now let $\Theta^R$ be the set of all $\theta \in \mathbb{R}^3$, such that $T(\theta) = \theta$. Obviously all $\theta \in \Theta^R$ are stationary points of the operator $T(\theta)$ and thus stationary points of the differential equation (6). Recall, that this means that the auxiliary model (3) coincides with one of the rational expectations equilibria (2-a) and (2-b) for all $\theta \in \Theta^R$. Looking at (12) and (13) we see that we can state the following proposition without any formal proof:[8]

**Proposition 1.** *Any* $\theta \in \Theta^R$ *is a stationary point of the differential equations (12) and (13).*

As already noted above, rational expectations equilibria are defined to be stationary points of the operator $T(\theta)$. Thus, proposition 1 states that any rational expectations equilibrium might be a possible outcome of the learning procedures considered here. Note that because $\Theta^*_{II}$ is a subset of $\Theta^R$, each of the differential equations (12) and (13) exhibits an unbouded continuum of stationary points.

The following proposition establishes the correspondence between the E-stability of rational expectations equilibria and the possible convergence of the LS algorithm:

**Proposition 2.** *A necessary and sufficient condition for* $\theta \in \Theta^R$ *and an associated* $R = M_z(\theta)$ *to be a (locally) stable stationary point of the differential equation (13) is that* $\theta$ *is a (locally)*

[8]For all $\theta \in D_\theta$ a stationary point of (13-b) is given by $R = M_z(\theta)$, such that this equation need not be considered.

*stable stationary point of the differential equation (6).*

The proof of this proposition is attributed to Marcet and Sargent (1989). It is based upon showing that for $\theta \in \Theta^R$ all eigenvalues of the Jacobian matrix of (13) that do not equal $-1$ coincide with the eigenvalues of $J(\theta)$ from (7): to check local stability it is sufficient to look at the subsystem (13-a). Since $T(\theta) = \theta$, for all $\theta \in \Theta^R$ the Jacobian matrix $J_{KQ}(\theta)$ results in $J_{KQ}(\theta) = R^{-1} M_z(\theta) J(\theta)$, where $J(\theta)$ is the Jacobian matrix from (7). Because a stationary point of (13) implies $R = M_z(\theta)$, we get $J_{KQ}(\theta) = J(\theta)$, such that the eigenvalues of $J_{KQ}(\theta)$ coincide with that of $J(\theta)$.

An immediate consequence of proposition 2 is that the LS algorithm will never converge to rational expectations equilibria that are not E-stable with respect to the auxiliary model. This in turn implies that in order to make predictions regarding the possible convergence of this learning procedure and in order to select among multiple equilibria it is sufficient to check whether the respective equilibria are E-stable or not. This is a quite comfortable result, because the formal proof of E-stability is much simpler to check than the possible convergence of the LS algorithm.

Thus, with respect to figure 1 and dependent upon the values of the parameters $\beta_1$ and $\beta_2$, we can draw the following conclusions about the LS algorithm: in region ①, the only E-stable equilibrium is (2-a) such that for all $\theta \in \Theta^R$ only $\theta = \theta^*_I$ is a possible convergence point for $\theta_t$ from (10). In region ②, $\theta_t$ will — if at all — converge only to $\theta \in \Theta^*_{II}$ that lead to a stationary ARMA solution, because only (2-b) is E-stable. Because none of these equilibria is E-stable in region ③, $\theta_t$ from (10) will not converge to a finite value in this case.[9]

However, this correspondence between the E-stability of a rational expectations equilibrium and the possible convergence of adaptive learning procedures does not generally hold. The result concerning the LS algorithm is due to the fact that So, for the SG algorithm (9), it is possible to show the following:

*(a) There exist $\theta \in \Theta^R$ which are (locally) stable stationary points of (12) that are not simultaneously (locally) stable stationary points of (6).*

*(b) The SG algorithm (9) may converge to equilibria that are not E-stable.*

[9] Note that formal proofs of this convergence are not possible, because the ARMA solutions (2-b) correspond to an unbounded continuum of stationary points of the differential equation (13-a). However, simultation results of Evans and Honkapohja (1994) verify that the convergence of the LS algorithm to an ARMA solution (2-b) is indeed dependent upon the E-stability of these equilibria. Moreover, as will be shown below, given the auxiliary model (3) even the white noise solution (2-a) corresponds to such an unbounded continuum of stationary points. For this case own simulation results that are not reported here showed that convergence of the LS algorithm to this solution occurs only if the condition of strong E-stability is met.

Statement (a) says that there is no correspondence between the stability of stationary points of (12) contained in $\Theta^R$ and the E-stability of the corresponding rational expectations equilibrium. Statement (b) uses the relationship between the differential equation (12) and the SG algorithm (9) to make a similar statement regarding the convergence of the SG algorithm. Thus, there exists an adaptive learning procedure that may converge to rational expectations equilibria which are not E-stable. This means that what is known about the LS algorithm cannot be generalized to other adaptive learning procedures and consequently, the concept of E-stability cannot be used without further qualification to judge the possible convergence of adaptive learning procedures.

In order to verify the above statements, it is first of all necessary to characterize the set of all stationary points of (12) and (13). Up to now, the analysis concentrated on the set $\Theta^R$ of the stationary points of $T(\theta)$, but this set is only a subset of the set of all stationary points, i.e. regarding the stationary points of (12) and (13) a restriction of the analysis only to stationary points of the operator $T(\theta)$ might be too strict. Let $\Theta^F$ be the set of all $\theta \in \mathbb{R}^3$, such that $M_z(\theta)\,[T(\theta)-\theta]=0$. Thus, $\Theta^F$ is the set of all stationary points of (12) and (13-a). We then have:

**Proposition 3.** *Define $\Theta^N = \Theta^F \setminus \Theta^R$. Then:*

$$\Theta^N = \left\{\theta\,|\,\theta' = ((1-\psi)\beta_0(1-\beta_1-\beta_2)^{-1}, \psi, -\psi), \quad \psi \in \mathbb{R}, \psi \neq 0\right\} \neq \{\emptyset\}.$$

*Proof.* Given $M_z(\theta)$ according to (11), the last equation in the system of equations $M_z(\theta)[T(\theta)-\theta] = 0$ reads as $(\psi+\phi)[\beta_1+\beta_2\psi-1]=0$. Therefore, its solutions are given by $\psi = -\phi$ and $\psi = \frac{1-\beta_1}{\beta_2}$, $\phi \in \mathbb{R}$. With $\phi = -\psi$, the second equation in this system of equations becomes:

$$\frac{T_\alpha(\theta)}{1-T_\psi(\theta)}[T_\alpha(\theta)-\alpha] + \left\{\left(\frac{T_\alpha(\theta)}{1-T_\psi(\theta)}\right)^2 + M_x\right\}[T_\psi(\theta)-\psi] + M_x\,[T_\phi(\theta)-\phi] = 0$$

Since $\phi = -\psi$, we have $T_\phi(\theta) = -T_\psi(\theta)$ and this equation is fulfilled, if $\alpha$ and $\psi$ satisfy the first equation

$$[T_\alpha(\theta)-\alpha] + \frac{T_\alpha(\theta)}{1-T_\psi(\theta)}[T_\psi(\theta)-\psi] = \frac{T_\alpha(1-\psi)}{1-T_\psi} - \alpha = 0$$

Substituting for $T_\alpha$ and $T_\psi$ from (4) then reveals that this requires $\alpha = (1-\psi)\,\frac{\beta_0}{1-\beta_1-\beta_2}$. □

Proposition 3 says that the differential equations (12) and (13) possess stationary points that are not simultaneously stationary points of the operator $T(\theta)$ and that do therefore not represent rational expectations equilibria in the usual sense.

Given a $\theta \in \Theta^N$, the general ARMA(1,1) solution of the model (1) becomes:

$$y_t = (1-\psi)\,\beta_0\,(1-\beta_1-\beta_2)^{-1} + \psi y_{t-1} - \psi x_{t-1} + x_t \tag{14}$$

As can be seen, this solution is generated from (2-a) by multiplication with the lag polynomial $(1-\psi\,\mathrm{L})$. Note that the process (14) and the solution (2-a) are equivalent for all $t$ only if $y_0 = x_0 + \frac{\beta_0}{1-\beta_1-\beta_2}$ (Evans and Honkapohja (1986)).[10] If this initial condition is not satisfied, both processes are at least asymptotically equivalent if $|\psi| < 1$, i.e if (14) is a stationary process. This means that expectations based upon (14) are not equivalent to rational expectations if the initial condition $y_0 = x_0 + \frac{\beta_0}{1-\beta_1-\beta_2}$ is not satisfied. However, these expectations will coincide asymptotically with rational expectations whenever $|\psi| < 1$.

Because $\theta$ is not a stationary point of (6) we have $T(\theta) \neq \theta$ for $\theta \in \Theta^N$. This implies that if the agents in the model form their expectations with respect to an ARMA(1,1) model parametrized by $\theta \in \Theta^N$ the resulting actual process for $y_t$ will not be given by (14). Substitution of $\theta \in \Theta^N$ into (8) reveals that the actual process for $y_t$ is given by:

$$y_t = \frac{\beta_0}{1-\beta_1-\beta_2}(1-\beta_1\psi-\beta_2\psi^2) + [\beta_1\psi+\beta_2\psi^2]\,y_{t-1} - [\beta_1\psi+\beta_2\psi^2]\,x_{t-1} + x_t \tag{15}$$

For any initial values, $y_0$ and $x_0$, the process (15) is asymptotically equivalent to (2-a) if $|\beta_1\,\psi + \beta_2\,\psi^2| < 1$.[11] This condition is equivalent to $|T_\psi(\theta)| < 1$, i.e. the stochastic process (8) is stationary for all $\theta \in \Theta^N$ that in addition satisfy $|\beta_1\psi+\beta_2\psi^2| < 1$. It should be noted that for this condition to hold it is neither necessary nor sufficient that $|\psi| < 1$. Since any $\theta \in \Theta^N$ that satisfies the condition $|\beta_1\psi+\beta_2\psi^2| < 1$ will lead at least asymptocically to expectations that coincide with rational expectations, I refer to such solutions as being asymptotic rational expectations equilibria and subsume these asymptotic rational expectations equilibria under the rational expectations equilibria of the underlying model.

Thus, the model (1) exhibits two distinct (asymptotic) rational expectations equilibria, each corresponding to a continuum of parameter vectors $\theta$: the set $\Theta^*_{II}$ from (5-b) generates ARMA solutions of the form (2-b) and the set

$$\Theta^*_I = \left\{\theta\,|\,\theta' = ((1-\psi)\beta_0(1-\beta_1-\beta_2)^{-1}, \psi, -\psi)\,, \quad |\beta_1\psi+\beta_2\psi^2| < 1\right\}$$

generates, at least asymptotically, the white noise solution (2-a).

Regarding the above made statements, the following things now have to be shown: first, it must be shown, that $\theta \in \Theta^*_I$ or $\theta \in \Theta^*_{II}$ may be stable stationary points of (12) even if the respective rational expectations equilibria (2-a) and (2-b) are not E-stable. Second, it must be shown that the SG algorithm will indeed converge to E-unstable equilibria. However, a

[10](14) implies $y_t = \frac{\beta_0}{1-\beta_1-\beta_2} + x_t + \psi^t\left(y_0 - \frac{\beta_0}{1-\beta_1-\beta_2} - x_0\right)$.

[11]From (15) we get $y_t = \frac{\beta_0}{1-\beta_1-\beta_2} + (\beta_1\psi+\beta_2\psi^2)^t\left[y_0 - \frac{\beta_0}{1-\beta_1-\beta_2} - x_0\right] + x_t$. If $y_0 \neq \frac{\beta_0}{1-\beta_1-\beta_2} + x_0$, $y_t$ will converge to the solution (2-a) for $t \to \infty$ only if $|\beta_1\psi+\beta_2\psi^2| < 1$.

formal proof of convergence is not possible, because $\Theta_I^*$ and $\Theta_{II}^*$ are two unbounded continua of stationary points. But even if this is disregarded, the formal analysis of the differential equation (12) is not a simple task: although it is indeed simple to compute the Jacobian matrix of (12) for $\theta \in \Theta^R$, this matrix will possess at least one eigenvalue that equals zero. This means that we cannot use standard techniques for stabilitiy analysis in this case. For $\theta \in \Theta^N$ even the computation of this Jacobian matrix is a very difficult task. For this reason, the remainder of this paper relies upon numerical simulations of the differential equation (12) and the SG algorithm (9) to verify the statements (a) and (b).

## 4.2 Simulation results

Let us begin with statement (a), which says that there are $\theta \in \Theta^R$ which are stable stationary points of (12), but which are not simultaneously stable stationary points of (6).

Figures 3 and 2 each show the results from solving the differential equation (12) numerically for alternative values of the parameters $\beta_1$ and $\beta_2$ and given fixed values for $\beta_0$ and $M_x$. The values for $\beta_1$ and $\beta_2$ are specified, such that for each configuration we have $|(1-\beta_1)/\beta_2| < 1$. Thus, the stochastic process (8) for $z_t$ will be stationary for all $\theta \in \Theta_{II}^*$. Stationarity of (8) for $\theta \in \Theta_I^*$ requires no further restrictions, because for all $\beta_1$, $\beta_2$ there exists a nonempty interval for $\psi$, implying stationarity.[12]

For these two specifications of the model, the respective differential equation was solved numerically given 25 randomly chosen initial vectors $\theta(0)$.[13] For all converging trajectories, the points in the figures show the resulting parameters $\psi(t)$ and $\phi(t)$ for $t = 100$.[14] The stopping time $t = 100$ was chosen large enough to interpret these points as (locally) stable stationary points of the respective differential equations. In all figures that follow, the parameters $\psi$ and $\phi$ of the auxiliary model (3) are plotted against each other. Thus, the two straight lines depicted in the figures each correspond to one of the continua of stationary points $\Theta_I^*$ and $\Theta_{II}^*$.

As can be seen from figure 2, given $\beta_1 = 1.25$ and $\beta_2 = -1.0$, there are two branches of (locally) stable stationary points: $\theta \in \Theta_I^*$ as well as $\theta \in \Theta_{II}^*$ are (locally) stable stationary

[12] If $\psi = 0$, we get $T_\psi(\theta) = 0$. Since $T_\psi(\theta)$ is quadratic in $\psi$ and takes an extreme value at $\psi = 0$, there exists an open interval around $\psi = 0$, where we have $|T_\psi(\theta)| < 1$. If $\beta_2 > 0$, we get $0 \leq T_\psi(\theta) < 1$ for all $\psi \in (\psi_u, \psi_o)$, where $\psi_u, \psi_o = \frac{1}{2\beta_2}\left\{-\beta_1 \pm \sqrt{\beta_1^2 + 4\beta_2}\right\}$; if $\beta_2 < 0$, we get $0 \geq T_\psi(\theta) > -1$ for all $\psi \in (\psi_u, \psi_o)$, where $\psi_u, \psi_o = \frac{1}{2\beta_2}\left\{-\beta_1 \pm \sqrt{\beta_1^2 - 4\beta_2}\right\}$.

[13] The numerical solutions were obtained with *Mathematica 2.2* using the function `NDSolve[ ]`.

[14] Although the random selection of the initial vectors was restricted to a close neighborhood of the stationary points in $\Theta^R$ and $\Theta^N$, the this procedure resulted quite frequently in diverging trajectories.

**Figure 2:** *Stable stationary points of the differential equation (12) with* $\beta_0 = 1$, $\beta_1 = 1.25$, $\beta_2 = -1.0$ *(corresponds to region ② in figure 1), and* $M_x = 1$ *(*$t = 100$*).*

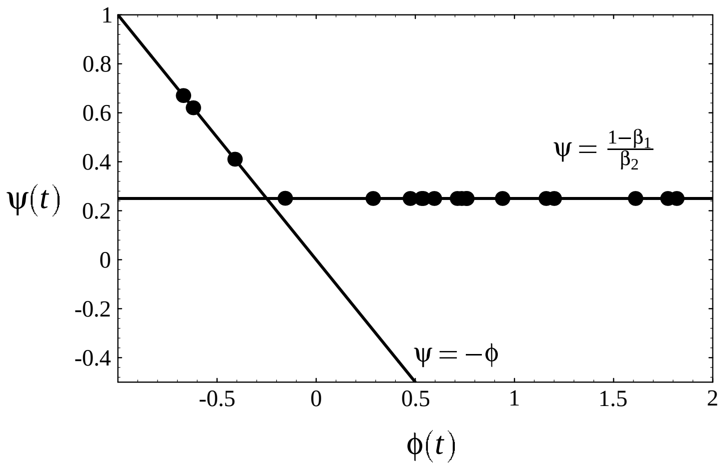
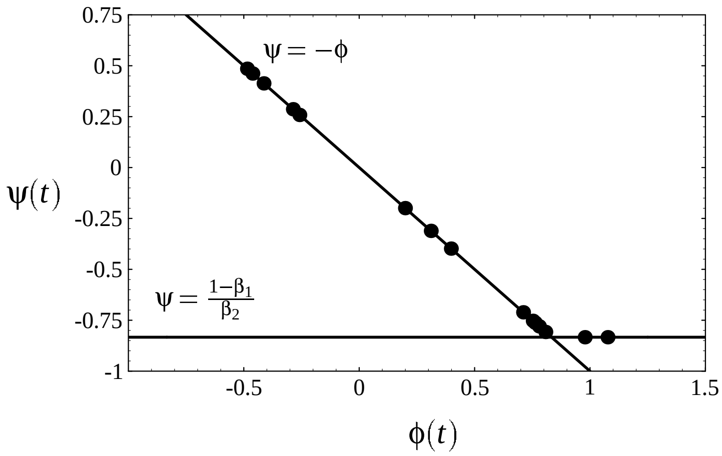



**Figure 3:** *Stable stationary points of the differential equation (12) with* $\beta_0 = 1$, $\beta_1 = -0.25$, $\beta_2 = -1.5$ *(corresponds to region ① in in figure 1), and* $M_x = 1$ *(*$t = 100$*).*




points of the respective differential equation.[15] This configuration implies that the white noise solution (2-a) is not strongly E-stable (however, it is still weakly E-stable). Although this results cannot be seen as a support of the statement (a) from above, because the stationary points in figure 2 that belong to $\Theta_I^*$ do not belong to $\Theta^R$, they lead at least to the conjecture that these stationary points and thus the white noise solution (2-a) might be the possible outcome of the SG algorithm.

Figure 3 shows a similar result for the other configuration of the model: if the parameters are given by $\beta_1 = -0.25$ and $\beta_2 = -1.5$, the ARMA solution (2-b) is not weakly E-stable, but there are $\theta \in \Theta_I^*$ as well as $\theta \in \Theta_{II}^*$ which are stable stationary points of the respective

[15] For this configuration, all $\psi \in (-0.554, 1.804)$ imply $|T_\psi(\theta)| < 1$.

**Figure 4:** *Convergence of the SG algorithm with* $\beta_0 = 1$, $\beta_1 = 1.25$, $\beta_2 = -1.0$ *(corresponds to region ② in figure 1), and* $x_t \sim N(0,1)$, $\kappa = 0.5$ *(*$t = 1, \ldots, 1000$*).*

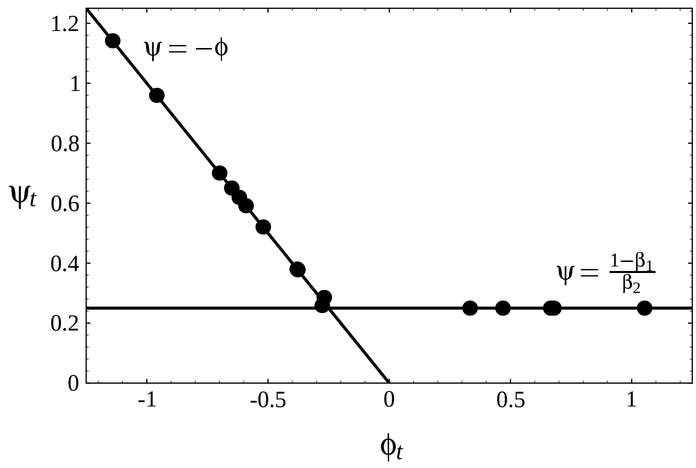
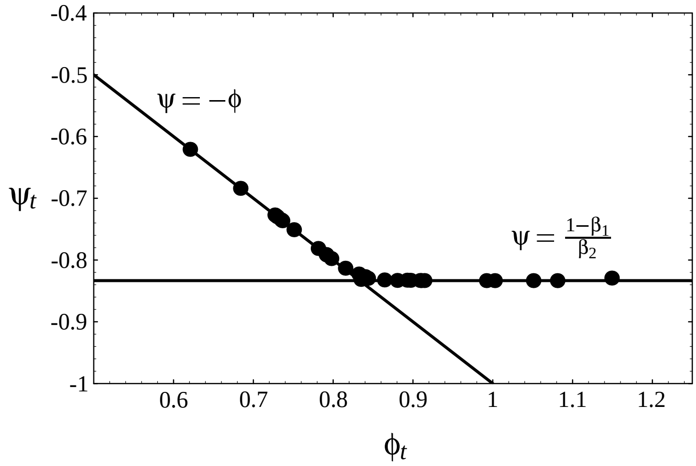



**Figure 5:** *Convergence of the SG algorithm with* $\beta_0 = 1$, $\beta_1 = -0.25$, $\beta_2 = -1.5$ *(corresponds to region ① in figure 1), and* $x_t \sim N(0,1)$, $\kappa = 0.5$ *(*$t = 1, \ldots, 1000$*).*




differential equation.[16] Thus, here we have indeed an example showing that result regarding the LS algorithm stated in proposition 2 does not hold for the SG algorithm.

Let us now turn to statement (b) from above. If the stability of stationary points in the differential equation (12) does not necessarily correspond to the E-stability of the respective rational expectations equilibrium, one might suppose that also the SG algorithm may converge to equilibria which are not (weakly or strongly) E-stable. Simulations of the SG algorithm show that this is indeed the case and verify the above made statement. These simualtions are based upon the same configuration of the parameters $\beta_1$ and $\beta_2$ used for the numerical solution of the associated differential equation. For both configurations, figures

[16]With $\beta_1 = -0.25$ and $\beta_2 = -1.5$, the stationarity of $z_t$, i.e. $|T_\psi(\theta)| < 1$, requires $\psi \in (-0.904, 0.737)$.

4 and 5 show the results of 120 simulations, each running over 1000 periods. The resulting parameter values for $\psi_t$ and $\phi_t$ after 1000 periods are depicted only for those cases where the SG algorithm converged. The respective initial values $\theta_0$ for these simulations where chosen randomly and with respect to the exogenous variable, it was assumed that $x_t$ is for all $t$ realization of a Gaussian random variable.

In figure 4 the points show the resulting parameter values for $\psi$ and $\phi$ for the configuration $\beta_1 = 1.25$ and $\beta_2 = -1.0$. As can be seen, the SG algorithm converges to solutions where $\psi = -\phi$, i.e. solutions that are asymptotically equivalent to the white noise solution (2-a). But because this white noise solution is not strongly E-stable (but it is nevertheless weakly E-stable), the results presented in figure 4 give an example of a situation where the possible convergence of an adaptive learning procedure does not correspond to the criterion of strong E-stability.

A similar result can be obtained with the other parameter configuration $\beta_1 = -0.25$ and $\beta_2 = -1.5$. Although the ARMA solutions (2-b) are not even weakly E-stable in this case, figure 5 reveals that the SG algorithm may nevertheless converge towards these ARMA solutions. Thus, here we have an example where even the criterion of weak E-stability does not help in order to assess the possible convergence of an adaptive learning procedure.

Both figures reveal that the SG algorithm may indeed converge to $\theta \in \Theta^N$, i.e. parameter vectors that are not stationary points of (6). Thus, it is by no means necessary that learning with an overparametrized model will result in values for the additional parameters that converge to zero, as it would be the case for the differential equation (6) based on the operator $T(\theta)$. While this result might be not very surprising it is nevertheless worth to be noted, because as shown in the figures, a simulated learning algorithm will quite often converge to such overparametrized solutions.[17]

In summary, the simulations of the SG algorithm allow the following conclusions: first, there exist solutions to the model (1) that are not strongly E-stable, but that are nevertheless possible solutions of an adaptive learning procedure. Second, there exist possible solutions of an adaptive learning procedure that are not even weakly E-stable. Third, there exist stationary points of the differential equation associated with the SG algorithm that are not stationary points of the operator $T(\theta)$ relevant for the concept of E-stability. These stationary points prove to be additional convergence points for the SG algorithm.

It should be noted that not every configuration of the parameters $\beta_1$ and $\beta_2$ allows for

[17] It should be noted that this behavior is not a special feature of the SG algorithm. Simulations performed with the LS algorithm have shown that this algorithm too quite often converges to $\theta \in \Theta^N$ (clearly, contrary to the SG algorithm this can only occur if the white noise solution (2-a) is strongly E-stable).

results as described above. For instance, any configuration that results in $|(1-\beta_1)/\beta_2| > 1$ implies that solutions of the form (2-b) are nonstationary. If, in addition, $\beta_1$ and $\beta_2$ are such that we are in region ① of figure 1, the consequence is that only $\theta \in \Theta_I^*$ — and therefore only the E-stable solution (2-a) — are possible convergence points for the SG algorithm. Because the intention of this paper is to show that even equilibria that are not E-stable may be the outcome of adaptive learning procedures, this causes no problems: the examples presented here are sufficient to substantiate this proposition.

# 5 Conclusions

The aim of this paper was to show that there may be no correspondence between the concept of E-stability and the possible convergence of adaptive learning procedures in linear economic models. Such a correspondence indeed exists if recursive least squares algorithms are taken into consideration. But these algorithms constitute merely one class out of many plausible adaptive learning procedures.

Assuming a simple linear rational expectations model and an alternative learning procedure, it has been shown that this learning procedure may converge to equilibria which are not (weakly or strongly) E-stable. This means that the connection between E-stability and the possible convergence of adaptive learning procedures is in general not as strong as that for the LS algorithm. However, to be fair, it must be emphasized that a general assertion saying that E-stability and the possible convergence of adaptive learning procedures correspond to each other is made nowhere in the literature. So, the message of this paper is merely that one should not hope for such a general assertion to be valid.

Surely, one might raise the objection that the LS algorithm is a more efficient estimation procedure than the SG algorithm considered here, such that the results of the paper are of little practical importance. However, the simulation results presented here show that the SG algorithm is a more robust estimation procedure, because contrary to the LS algorithm it may converge even to equilibria which are not E-stable. But whenever it is possible to use the criterion of efficiency in order to select among various adaptive learning procedures it is by no means inept to use the alternative criterion of robustness.

By the way, the paper also gives another example of the problems that are associated with the selection of equilibria by means of a 'learnability' criterion. While the LS algorithm gives a definite result in that at most one of the two equilibria of the model would be selected, this is not true for the SG algorithm. As has been indicated by the simulation results shown above (and some more that are not reported here), the SG algorithm would not select a unique equilibrium whenever the ARMA solutions (2-b) are stationary and the white noise solution (2-a) is at least weakly E-stable. Thus, apart from the fact that it might not be possible to select a unique equilibrium using such a 'learnability' criterion, there might exist different but equally plausible learning procedures that each select different equilibria.

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