Growth and Labor Income Risk with Inelastic and Elastic Labor Supply

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Abstract

We discuss long–run growth in an economy which is subject to aggregate productivity shocks affecting all factors of production. We demonstrate that the presence of labor income risk unambiguously is an important determinant of long–run expected growth. The issue of dynamic inefficiency of the underlying allocation is related to the size of the risk premium on capital return. The paper also examines the effects distributive disturbances and elastic labor supply, the latter giving rise to the possibility of multiple equilibria.

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1 Introduction

The importance of capital and income risk for the intertemporal savings decision of risk averse agents was first addressed by Sandmo (1970) in a two–period expected utility setting. The focus lies especially on precautionary savings, which, referring to pioneering work of Leland (1968), can be defined as savings a risk averse agent additionally undertakes in order to self–insure against the riskiness of future income flows.

Only recently, this specific side of intertemporal decision making has gained new attraction in modern growth theory. Yet, the related literature on this topic suffers from the important shortcoming that most contributions restrict their analysis to a single type of income, either capital risk or income risk, in order to maintain analytical tractability; see Eaton (1981), Weil (1993), Obstfeld (1994), and Smith (1996). An alternative perspective, propagated for instance by Femminis (1995) and Corsetti (1997), is to view the intertemporal flow of labor incomes as human wealth and treat it as a 'quasi accumulating' and hedgeable asset. The present paper provides an attempt to close this gap by examining the impact of labor and capital risk on long–run expected growth within one consistent continuous–time stochastic general equilibrium framework.

A different body of literature discusses labor supply in the context of the stochastic neoclassical growth model but is not primarily concerned with the consequences of stochastic labor incomes for individual intertemporal decision–making, see e. g. Bourguignon (1974) and Merton (1975). Amilon and Bermin (2003) extend these contributions to a stochastic Ramsey model to analyze the welfare effects of controlling labor supply, where capital accumulation is assumed to be instantaneously deterministic and labor force dynamics are governed by a stochastic process.

The neoclassical stochastic growth model also is the workhorse of real business theory. Here, labor supply usually is assumed to be elastic, the analysis focusing on co–movements of macroeconomic aggregates over the business cycle and on the response to temporary shocks; see King and Rebelo (1999). The relation between elastic labor supply and long–run growth has also been examined in some depth in deterministic models, see for instance Eriksson (1996), Ladrón-de Guevara *et al.* (1997), or lately Ortigueira (2000), de Hek (1998, 1999) and Turnovsky (1999, 2000*a*). Extensions to a stochastic context can be found in Bodie *et al.* (1992), Basak (1999) and Turnovsky (2000*b*, 2003).

Bodie *et al.* (1992) incorporate labor supply flexibility in a life cycle model but do not address issues of macroeconomic growth. They find that labor supply flexibility smoothes the intertemporal consumption flow, by creating an insurance against adverse investment outcomes on markets for financial assets. Basak (1999) examines an endogenous labor–leisure choice in a stochastic consumption–based capital asset pricing model. Similarly to Bodie *et al.* (1992) and Femminis (1995), the author treats human wealth (i. e. the present value of future labor incomes) as a priceable and hedgeable asset, but, unfortunately, abstracts from physical capital in the underlying stochastic production technology. Turnovsky (2000*b*, 2003) discusses the other side of the coin, assuming that returns to capital absorb the entire production risk and wages are nonstochastic.

The model we are going to present in this paper combines both, uncertain wage incomes as well as stochastic returns to physical capital. We demonstrate that this assumption has important consequences for the expected growth path of the economy. The major goal of our analysis is to explore several dimensions of how factor income risk interacts with long–run growth. On the labor supply side, we compare the implications of inelastic relative to elastic labor supply for the balanced growth path. Regarding the nature of the underlying disturbances, for most of the time, we follow the standard approach of real business cycle theory and assume aggregate productivity shocks. Since, in this case, factor returns are perfectly correlated with output, we extend the model by also allowing for distributive disturbances which are correlated with the output shock. We are interested in how these shocks affect economic growth and to what extent they improve the explanation of the empirically observed correlations between the macroeconomic variables.

All models presented here are extended continuous–time stochastic versions of the Romer (1986) endogenous growth model with human capital externalities, where the random disturbance stems from an aggregate productivity shock. We have chosen this framework basically for two reasons: On the one hand, the underlying production technology employs capital and labor as private factors of production on a constant return to scale basis. Inputs are payed according to their marginal products, which, being stochastic too, give rise to the desired capital and labor income risk, our analysis primarily focuses on. The economy evolves according to a stochastic trend, where the impact of risk on accumulation is reflected by second–order effects from the variance of the productivity shock in the long–run expected growth rate.[1] On the other hand, the learning–by–doing approach displays the convenient feature that the aggregate production is linear in capital,

[1]Technically, our model is closest to Turnovsky (2000*b*, 2003).

thereby sustaining ongoing growth, and, moreover and similarly important in a stochastic context, it enables us to derive a closed–form solution for the model.

The following section 2 gives a review of the benchmark model with inelastic labor supply. The main purpose of this section is to introduce the basic setting common to the subsequent model extensions and to provide the reader with a first intuition of how the riskiness of factor incomes relates to growth. Section 2 also contrasts the equilibrium allocation of the competitive economy with the Pareto–efficient one and relates the emergence of either suboptimally low growth or dynamically inefficient (i. e. suboptimally high) growth to the size of the risk premium. Section 3 examines the effects of distributive shocks, thereby adding a second source of uncertainty to the model and taking account of the empirical observation that factor returns are not perfectly correlated with aggregate output. Section 4 then returns to the benchmark setting, while relaxing the assumption of an inelastic labor supply. Regarding the issue of existence and uniqueness of a macroeconomic equilibrium, we show that endogenizing the labor–leisure choice in the stochastic environment gives rise to the possibility of both, a unique one as well as multiple equilibria, the outcome crucially depending on whether or not the growth process is characterized by dynamic inefficiency. Section 5 concludes.

2 Benchmark: Inelastic Labor Supply

The model We assume an economy populated by a continuum $[0,1]$ of identical, infinitely–lived agents. The agents are homogeneous with respect to their preferences and their ex ante endowments. Each individual selects the optimal amount of consumption and savings in order to maximize intertemporal welfare according to the additively separable von Neumann–Morgenstern expected utility function

$$V(0) = \mathrm{E}_0 \int_0^\infty U[c(t)]\, e^{-\beta t}\ \mathrm{d}t. \tag{1}$$

Here, $c(t)$ is the intertemporal consumption flow which we assume to be instantaneously deterministic. The parameter $\beta > 0$ is the subjective rate of time preference. E_0 is the expectations operator conditional on time–0 information. The current period utility function $U[c(t)]$ is assumed to be isoelastic and strictly concave with a positive third derivative

$$U[c(t)] = \frac{c(t)^{1-\rho}}{1-\rho} \quad \text{if} \quad \rho > 0, \rho \neq 1, \tag{2}$$

and $\ln c(t)$ if $\rho = 1$. Current period utility displays constant relative risk aversion, denoted by the risk aversion index ρ.

Individual production is stochastic. At each increment of time the economy is subject to an aggregate productivity shock. The representative firm produces a homogeneous good according to the following stochastic Cobb–Douglas–technology

$$\mathrm{d}y(t) = Ak(t)^{\alpha}L(t)^{\gamma}K(t)^{1-\alpha}\left(\mathrm{d}t + \mathrm{d}z(t)\right), \qquad \alpha,\gamma \in (0,1) \ . \tag{3}$$

$\mathrm{d}z(t)$ is the increment to a standard Wiener process $z(t)$ with zero mean and the instantaneous variance of production $\sigma^2\,\mathrm{d}t$. We apply the learning–by–doing setting introduced by Romer (1986). The instantaneous output flow $\mathrm{d}y(t)$ is assumed to be generated from physical capital $k(t)$ and labor $L(t)$. For expository purposes only, we denote the factor income shares of the competitive equilibrium with α and γ in order to emphasize the effect stemming from the respective source of income, although, of course, we assume production to be linearly homogeneous on the firm level. With constant returns to scale, that is $\alpha+\gamma = 1$, the requirements for a long–run competitive equilibrium are fulfilled.

We take labor to be inelastically supplied, and the labor force is normalized to unity. Nevertheless, the household receives an income from both factors of production. In terms of Sandmo (1970), the agent is exposed to *capital risk* and *income risk*. The diffusion processes for capital and wage incomes over the time increment $(t, t+\mathrm{d}t)$ are given by

$$\begin{aligned} \mathrm{d}P(t) &= r(t)\,k(t)\;\mathrm{d}t + \mathrm{d}z_P(t)\ , \\ \mathrm{d}W(t) &= w(t)\,L(t)\;\mathrm{d}t + \mathrm{d}z_W(t)\ , \end{aligned} \tag{4}$$

where $r(t)$ is the expected rate of return to capital, $w(t)$ is the expected wage rate, such that $r(t)\,k(t)$ and $w(t)\,L(t)$ denote the instantaneous drift rates of the incomes processes, whereas $\mathrm{d}z_P(t), \mathrm{d}z_W(t)$ represent the respective instantaneous stochastic components. All are to be determined in equilibrium.

The production function (3) exhibits human capital externalities. The aggregate stock of technical knowledge, $K(t)$, acts as a Harrod–neutral growth parameter and is enhanced by investment in privately owned capital. In equilibrium, $r(t)$ equals the private marginal product of capital and falls short of the social return. $k(t)$ equals $K(t)$ in macroeconomic equilibrium, due to the normalization of population size, and aggregate production is linear in capital. Hence, the requirements for ongoing growth of per capita incomes are met. This assumption together with the assumption stated on the nature of the random disturbance implies that the economy evolves according to a stochastic trend.

Individuals save by investing in risky physical capital. For analytical simplicity, we assume the capital stock to depreciate completely at the end of each time increment such that investments equal the future capital stock. The representative agent is endowed with the initial capital stock $k(0) > 0$. In each time increment t, she receives capital and labor incomes. Her flow budget constraint is then given by

$$\mathrm{d}k(t) = \mathrm{d}W(t) + \mathrm{d}P(t) - c(t)\,\mathrm{d}t = [r(t)\,k(t) + w(t)\,L(t) - c(t)]\,\mathrm{d}t + \mathrm{d}z_k(t)\ . \quad (5)$$

The diffusion process of capital is given by $\mathrm{d}z_k(t) = \mathrm{d}z_P(t) + \mathrm{d}z_W(t)$ and the associated instantaneous variance of physical capital can be derived as

$$\sigma_k^2(t) = \frac{\mathrm{E}\,[\mathrm{d}z_k(t)]^2}{\mathrm{d}t} = \sigma_P^2(t) + \sigma_W^2(t) + 2\sigma_{PW}(t)\ , \quad (6)$$

where $\sigma_{PW}(t) = \rho_{PW}(t)\,\sigma_P(t)\,\sigma_W(t)$ denotes the instantaneous covariance between capital and labor incomes and $\rho_{PW}(t)$ represents the instantaneous correlation coefficient.

Household optimization The consumer's problem is to select her rate of consumption to maximize lifetime utility (1) subject to the budget constraint (5), and $k(0) > 0$, $z(0) = 0$, while taking factor prices as given. The stochastic Hamiltonian can be set up as follows[2]

$$\max_c \quad H\left(c, k, \lambda, \frac{\partial\lambda}{\partial k}\right) = U(c)\,e^{-\beta t} + \lambda\,[rk + wL - c] + \frac{\sigma_k^2}{2}\,\frac{\partial\lambda}{\partial k}\ . \quad (7a)$$

The first–order conditions are

$$\frac{\partial H}{\partial c} = U_c\,e^{-\beta t} - \lambda = 0 \quad (7b)$$

$$\mathrm{d}\lambda = -\frac{\partial H}{\partial k}\,\mathrm{d}t + \frac{\partial\lambda}{\partial k}\,\mathrm{d}z_k = -\left[\lambda r + \frac{1}{2}\,\frac{\partial\lambda}{\partial k}\,\frac{\partial\sigma_k^2}{\partial k}\right]\mathrm{d}t + \frac{\partial\lambda}{\partial k}\,dz_k\ . \quad (7c)$$

together with the transversality condition

$$\lim_{t\to\infty}\ \mathrm{E}_t[\lambda(t)\,k(t)] = 0\ . \quad (7d)$$

Application of Itô's Lemma in order to differentiate (7b) with respect to time yields the following expression for the stochastic evolution of the shadow price over time

$$\mathrm{d}\lambda = e^{-\beta t}\left[U_{cc}\,\mathrm{d}c + \tfrac{1}{2}\,U_{ccc}\,(\mathrm{d}c)^2 - \beta U_c\,\mathrm{d}t\right]\ . \quad (8)$$

[2]Note that, in what follows, we drop the explicit time notation (t) for expository convenience.

Taking account of the nature of an optimal policy function $c(k)$, consumption c itself is stochastic. The diffusion $\mathrm{d}c$ can then be obtained by the Taylor expansion

$$\mathrm{d}c(k) = c'(k)\,\mathrm{d}k + \tfrac{1}{2}c''(k)(\mathrm{d}k)^2\,. \tag{9}$$

The solution conjecture usually applied for isoelastic preferences is that the propensity to consume out of capital, $\mu = c/k$, is constant in macroeconomic equilibrium. Hence, $c'(k) = \mu$ and $c''(k) = 0$. Substitution of (9) into (8), taking account of (5) and the Itô multiplication rules, substituting (7b) into (7c) and finally equalizing (8) to (7c) yields

$$\begin{aligned} e^{-\beta t}\left[\mu(rk + w - \mu k)U_{cc} + (r-\beta)U_c + \frac{\mu^2\sigma_k^2}{2}U_{ccc} + \frac{1}{2}\frac{\partial\lambda}{\partial k}\frac{\partial\sigma_k^2}{\partial k}\right]\mathrm{d}t = \\ \left[\frac{\partial\lambda}{\partial k} - e^{-\beta t}\mu U_{cc}\right]\mathrm{d}z_k\,. \end{aligned} \tag{10}$$

Substitution of the derivatives U_c, U_{cc}, U_{ccc} of instantaneous utility (2) into (10), and taking into consideration that the solution conjecture of a non-stochastic consumption–capital ratio is only satisfied if the following condition holds for the costate variable λ

$$\frac{\partial\lambda}{\partial k} = e^{-\beta t}\mu U_{cc}\,, \tag{11}$$

thereby canceling out the diffusion process on the RHS of (10), then rearranging, this finally leads to the following expression for the propensity to consume

$$\mu = \frac{1}{\rho}\left[\beta + (\rho-1)r\right] + \frac{w}{k} - \frac{1}{2}(\rho+1)\frac{\sigma_k^2}{k^2} + \frac{\partial\sigma_k^2}{\partial k}\,. \tag{12}$$

The consumption–capital ratio depends on the rates w and r, to be derived in market equilibrium, which, by (6), also determine the variance of the capital stock.

Market equilibrium The aggregate technology shock in the production function (3) is the single source of uncertainty in the economy. It affects the marginal products of the two private factors of production, thereby being entirely responsible for the randomness of labor and capital incomes. Given the technology (3), the equilibrium values of factor prices can be obtained by the usual marginal productivity conditions of the firm problem, which, after substitution in (4) and taking account of the equilibrium condition $k = K$ as well as the normalization $L = 1$, implies that the instantaneous drift and diffusion rates of the incomes processes are proportional to the current state,

$$\begin{aligned} \mathrm{d}P &= P(\mathrm{d}t + \mathrm{d}z) = \alpha A K(\mathrm{d}t + \mathrm{d}z)\ , \\ \mathrm{d}W &= W(\mathrm{d}t + \mathrm{d}z) = \gamma A K\,(\mathrm{d}t + \mathrm{d}z)\ . \end{aligned} \tag{13}$$

Similar to the corresponding values of the deterministic model, the expected rate of return to physical capital, $r = \alpha A$, is constant in macroeconomic equilibrium, while the expected wage rate, $w = \gamma AK$, rate grows linearly in the aggregate capital stock. The factor income shares $P/Y = \alpha$ and $W/Y = \gamma$ are nonstochastic and equal the respective partial elasticities of production, which is a typical result for the underlying Cobb–Douglas–technology.

Combining the intertemporal budget constraint (5) with the market prices (13) shows that the intertemporal consumption path indeed is characterized by a time–invariant relationship between consumption and wealth, expressed in the propensity to consume out of capital:[3]

$$\mu = \tfrac{1}{\rho}\left[\beta + (\rho - 1)\,\alpha A\right] + \gamma A + \tfrac{1}{2}\sigma^2 A^2\left[\alpha(1-\rho) - \gamma(\rho+1)\right]\ . \tag{14}$$

Substituting the equilibrium value of the propensity to consume (14) into the market clearing condition

$$\mathrm{d}K = \mathrm{d}Y - C\,\mathrm{d}t\ , \tag{15}$$

leads to the stochastic growth rate of the economy

$$\tilde{\psi} = \frac{\mathrm{d}K}{K} = \psi\,\mathrm{d}t + A\,\mathrm{d}z\ , \tag{16}$$

where, given that $\mathrm{E}[\mathrm{d}z] = 0$, the expected growth rate $\psi = \mathrm{E}(\mathrm{d}K)/(K\,\mathrm{d}t)$ can be determined as a closed–form expression in the primitives only

$$\psi = \tfrac{1}{\rho}(\alpha A - \beta) + \tfrac{1}{2}\,\sigma^2 A^2\left[\alpha(\rho - 1) + \gamma(\rho + 1)\right]\ . \tag{17}$$

Equation (17) shows that productivity shocks impact on expected growth via second–order effects stemming from the variance of the technological disturbance, thereby underlining the argument from above that economic development follows a stochastic trend.

The time–0 expected value of maximized lifetime utility can be derived by utilizing $c = \mu k$ and the equilibrium expected growth rate (20), substituting (14) into (1), determining the $(1-\rho)$–th moment of the capital stock,

[3] We employ the condition $\alpha+\gamma = 1$, whenever the sum $\alpha+\gamma$ appears explicitly in the equilibrium conditions of the macroeconomic variables. Without this simplification, the propensity to consume would write as: $\mu = \frac{1}{\rho}\left[\beta + (\rho-1)\,\alpha A\right] + \gamma A + \frac{1}{2}(\alpha+\gamma)\sigma^2 A^2\left[\alpha\,(1-\rho) - \gamma(\rho+1)\right]$.

by taking into consideration that the underlying productivity shock is log–normally distributed over time. Finally integrating (1) yields:

$$V(0) = \frac{[\mu k(0)]^{1-\rho}}{(1-\rho)\,\left[\beta-(1-\rho)\left(\psi-\frac{1}{2}\rho A^2\sigma^2\right)\right]}\,, \tag{18}$$

and $V(0) = \left(\beta\ln\mu+\beta\ln k(0)+\psi-\frac{1}{2}A^2\sigma^2\right)/\beta^2$ for logarithmic preferences respectively. Lifetime utility is bounded if the transversality condition $\lim\limits_{t\to\infty}\mathrm{E}_t\left[\lambda(t)\,k(t)\right]=0$ is satisfied, which is the case if the second term in the denominator of (18) is positive, that is, for

$$\beta-(1-\rho)\left[\psi-\tfrac{1}{2}\rho A^2\sigma^2\right]>0\,. \tag{19}$$

Variances and covariances in macroeconomic equilibrium In equilibrium, all macroeconomic aggregates grow at the common stochastic rate (16). For this reason the variances and covariances of the macroeconomic equilibrium are quite easy to assess. By application of the Itô multiplication rules and taking account of (4) and (13), it is possible to obtain the moments $\mathrm{E}(\mathrm{d}x)^2/\mathrm{d}t=\sigma_x^2$, $\mathrm{E}[\mathrm{d}z_x\,\mathrm{d}z_y]/\mathrm{d}t=\sigma_{xy}$ for output, consumption, investment, factor incomes and the stochastic growth rate. The standard deviations and associated coefficients of correlation can then be summarized as follows:

$$\begin{gathered}\sigma_Y=\sigma_K=AK\sigma \quad \sigma_P=\alpha\sigma_Y\,, \quad \sigma_W=\gamma\sigma_Y\,, \quad \sigma_C=\mu\sigma_Y\,, \quad \sigma_{\tilde{\psi}}=A\sigma\\ \rho_{YC}=\rho_{YK}=\rho_{YW}=\rho_{YP}=\rho_{PW}=1\,.\end{gathered} \tag{20}$$

The standard deviations of output, consumption as well as profit and labor incomes grow linearly in the capital stock, which, once more, demonstrates that the economy evolves according to a stochastic trend, where the standard deviation of the growth rate is stationary. We see that aggregate consumption is less volatile than aggregate income, the scaling factor being the propensity to consume out of capital. This reflects the standard result of life cycle/permanent income–theory stating, that the households try to smooth consumption by transferring irregular income flows into regular consumption flows via accumulation.

Moreover, (20) reveals that the income variables and consumption are perfectly correlated, which gives the model an unrealistic touch from the viewpoint of business cycle empirics. There, findings suggest investment to be more volatile than output, which again displays a larger volatility than consumption and the aggregate capital stock.[4] Following the analysis of

[4]Recall that we assumed the capital stock to depreciate completely at the end of each period, which for itself is a sufficient reason enough to explain why our model does not match the data related to the volatility of investment.

Heinemann (1995) for German (West) data, the correlation of output with consumption should center around $\rho_{YC} \approx 0.8$, whereas the correlation of output and investment amounts to $\rho_{YK} \approx 0.9$. The wage rate should be moderately procyclical, while lagging 3–5 quarters to GDP, with a maximum correlation of $\rho_{YW} \approx 0.73$. Similar results have been obtained for the U.S. economy; see Kydland and Prescott (1991).

Growth effects of capital and income risk Equations (14) and (17) show that the propensity to consume out of capital as well as the expected growth rate of the economy are the sum of two components, the first being equal to the associated value of the variables under certainty, the second measuring the consequences of the stochastic environment for growth. If we now tie the impact of the income sources to the respective factor income share, α, γ, obviously, the two income types are relevant in both components of the expected propensity to consume, the certainty equivalent as well as in the higher–order term. Non–surprisingly, the labor income share is not included in the certainty equivalent of the expected growth rate, which satisfies the common economic intuition from deterministic settings that the income from non–accumulating factors does not affect the long–run growth rate of the economy. Nevertheless, besides α, the labor income share γ is part of the second component of the expected growth rate.

Let us now start our analysis with taking a look at the propensity to consume out of wealth given by (14). If we recall the above statements on consumption smoothing in the discussion of the standard deviations and correlations in macroeconomic equilibrium (see eq. (20)), we find that the larger μ, the more volatile is the consumption flow of a representative agent for a given volatility of income. Then, changes in the factor associated risks tending to reduce the propensity to consume have a smoothing effect on consumption.

Since we are interested in these factor specific incomes risks, we discuss a variation of α and γ respectively, while holding the other income share fixed.[5] The response of μ to a variation in the income shares can then be

[5]We choose this strategy for expository purposes. It is understood that, with constant returns to scale in production, an increase in the capital income share is to be accompanied by a proportional decrease in the labor income share and vice versa in order to support a competitive equilibrium.

determined as follows:

$$\frac{\partial \mu}{\partial \gamma} = A - \tfrac{1}{2}\,(\rho+1)\,A^2\sigma^2\,, \tag{21a}$$

$$\frac{\partial \mu}{\partial \alpha} = (\rho-1)\,A - \tfrac{1}{2}\,(\rho-1)\,A^2\sigma^2\,. \tag{21b}$$

The first term in (21a) stands for the well–known wealth effect on consumption, which is positive, whenever current and future consumption are normal goods. Other things equal, a rise in the labor income share represents an increase in human wealth, thereby raising the level of consumption at all instants of time. Contrary, a rise in γ also reflects the situation that a larger fraction of the consumer's budget stems from risky wage incomes. The second term of (21a) unambiguously is negative, thereby reflecting Sandmo's results on the emergence of precautionary savings in the presence of income risk (Sandmo, 1970).[6] The desire to smooth the intertemporal consumption flow via additional savings is intensified the more risk averse the consumer or the larger the volatility of income. Both factors decide upon which of the incentives dominates in the end, either the positive wealth effect or the negative precautionary effect.

A change in the capital income share affects the equilibrium value of the real interest rate. In this situation, economic theory predicts counteracting intertemporal income and substitution effects. The income effect is positive in the first term of (21b). Because a rise in the interest rate allows higher consumption in the future for a given present value of lifetime resources, the expansion of the feasible consumption set induces the household to raise present consumption. Contrary, the substitution effect is negative in the first term of (21b), indicating that, as opportunity costs increase, savings become more attractive and the propensity to consume declines. The signs of both effects reverse in the second term of (21b). An increase in capital risk raises the mean as well as the volatility of future income flows, such that now the intertemporal income effect becomes negative, representing the importance of the precautionary motive for saving. The positive intertemporal substitution effect reflects the consumer's response to the rise in volatility. Since she dislikes future resources being exposed to growing risk, present consumption is raised and savings decline. In general we observe the well–known result from the analysis of capital risk, which states that the intertemporal income

[6] Sandmo (1970) shows that a decreasing absolute risk aversion is a necessary and sufficient condition for a household to save out of precautionary motives in the presence of a *pure* income risk. The instantaneous felicity function assumed in (2) satisfies this requirement for all degrees of risk aversion $\rho > 0$.

effect dominates the intertemporal substitution effect in (21b), whenever the coefficient of risk aversion exceeds unity (see Levhari and Srinivasan, 1969; Sandmo, 1970). Analogously, the substitution effect dominates for $\rho < 1$ and both effects exactly offset for $\rho = 1$ (logarithmic preferences).

If we now turn towards the expected growth rate of the economy, it becomes evident from the market clearing condition (15) that the terms measuring the consumer's response to the riskiness of future income flows in the propensity to consume, μ, also appear in the expected growth rate, but with opposite sign. Differentiating ψ with respect to γ and α yields:

$$\frac{\partial\psi}{\partial\gamma} = \frac{1}{2}A^2\sigma^2(\rho+1) > 0\,, \tag{22a}$$

$$\frac{\partial\psi}{\partial\alpha} = \frac{A}{\rho} + \frac{1}{2}A^2\sigma^2\,(\rho-1)\,. \tag{22b}$$

The expected growth rate unambiguously increases with a rise in the labor income share, which reflects the household's precautionary motive in the presence of income risk, thereby contradicting the usual notion from deterministic environments that non–accumulating factors have no impact on long–run growth. The positive first term in (22b) shows the standard result for riskless economies that an increase in capital return raises the attractiveness of savings, thereby promoting growth, while the second expression reflects the results from the discussion of income and substitution effects above. Only when the consumer is sufficiently risk averse, she will save out of precautionary motives in the presence of capital risk.

What else can we say about the interaction of alternative factor incomes in the determination of the expected growth rate? First, if we return to the equilibrium condition (17), we see that precautionary savings may be observed even for comparably low degrees of risk aversion, as long the second term related to labor income risk together with the positive intertemporal income effect more than compensates the negative substitution effect of capital risk on growth. This observation questions especially the certainty equivalent argument, often brought forward in connection with the assumption of logarithmic preferences, where, for a *pure* capital risk, the intertemporal effects offset and the risk–related expression in the expected growth rate vanishes. In our model, the parametric threshold for the coefficient of risk aversion associated with certainty equivalence is shifted downwards. The higher the labor income share, the lower is the critical value for the coefficient of risk aversion to sustain precautionary savings.

Second, if we take account of the fact that $\alpha + \gamma = 1$, which means that each increase in one income share is to be accompanied by an equivalent decrease in the other, we are able to demonstrate that this proportionality argument does not hold for the resulting growth effects:

$$\frac{\partial\psi}{\partial\alpha} = \frac{A}{\rho} + \frac{1}{2}A^2\sigma^2(\rho-1) + \frac{\partial\psi}{\partial\gamma} \times \frac{\partial\gamma}{\partial\alpha} = \frac{A}{\rho}\left(1-\rho A\sigma^2\right). \tag{23}$$

The effects stemming from an increase in the capital income share are not proportionately offset by the ones resulting from a decline in the labor income share. The risk–related effect is negative, while the overall effect from raising α is positive as long as the certainty equivalent to capital return is positive, which can implicitly be determined from the optimality condition (7c) by substituting the equilibrium interest rate (13), conditions (7b) and (11) into (7c) and taking expectations

$$r_s = \alpha A\left(1-\rho A\sigma^2\right), \tag{24}$$

and will also be important, when we contrast the allocation of the competitive economy with the Pareto–efficient one. The certainty equivalent to capital return can be viewed as the market clearing interest rate of a (ficticious) riskless asset, which, by the assumption of homogeneous agents, is not available here. The rate r_s is determined by the mean rate of return to capital αA and by the risk premium $\rho\alpha A^2\sigma^2$, which the risk averse agents demand for bearing the investment risk.

The efficient allocation It is a well–known feature of the Romer (1986) approach of modeling endogenous growth that, in the presence of human capital externalities, there is a wedge between private and social returns to capital. Consider now a benevolent social planner who maximizes the representative agent's intertemporal welfare (1), while taking account of this distortion

$$\begin{aligned} \max_{C} \quad & V(0) = \mathrm{E}_0 \int_0^\infty U(C)\,e^{-\beta t}\,\mathrm{d}t \\ \text{s. t.} \quad & \mathrm{d}K = [A-\mu^*]K\,\mathrm{d}t + AK\,\mathrm{d}z \end{aligned} \tag{25}$$

and $K(0) > 0$, $z(0) = 0$. By following the procedure described above, the Pareto–efficient values of the macroeconomic variables can be derived as:

$$\mu^* = \tfrac{1}{\rho}\left[\beta + (\rho-1)A\left(1-\tfrac{1}{2}\rho A\sigma^2\right)\right], \tag{26}$$

$$\psi^* = \tfrac{1}{\rho}(A-\beta) + \tfrac{1}{2}(\rho-1)A^2\sigma^2. \tag{27}$$

Apparently, the social planner is not expected to take care of the riskiness of a specific income source, and chooses the intertemporal consumption path such that the spillover effects are internalized and, roughly speaking, capital is payed its social return. The resulting allocation is equivalent to the one derived for a simple linear *AK*–technology (cf. Clemens and Soretz, 1999). In the deterministic setting, this implies a higher growth rate than in the competitive equilibrium. This result does not necessarily extend to the stochastic context. On the one hand, the agents of the competitive economy underestimate the mean return to capital. The certainty equivalent part of the expected growth rate (17) expresses these insufficient incentives to accumulate by being smaller than its efficient counterpart in (27). This outcome is in accordance with the deterministic model. On the other hand, by focusing solely on the riskiness of labor incomes and the private return to capital, the households of the competitive economy also underestimate the volatility of the capital stock. This becomes obvious if we compare the risk–related parts of the two expected growth rates (17) and (27). In (17), the risk–induced accumulation is too large, which means that a correct perception of the volatility of capital returns would induce a larger intertemporal substitution effect, thereby lowering the expected growth rate.

This effect of saving too much out of precautionary motives may lead to a situation characterized by even suboptimally high growth, where the expected growth rate of the market economy exceeds the Pareto–efficient one (cf. Smith, 1996; Clemens and Soretz, 1997; Clemens, 2002). It is possible to relate the emergence of dynamically inefficient allocations to the certainty equivalent to capital return as given by (24). We find:

Proposition 1 *A non–negative certainty equivalent to capital return is a necessary and sufficient condition for the expected growth rate of the market equilibrium not to exceed the expected Pareto–efficient growth rate:*

$$\psi \lesseqgtr \psi^* \quad \Longleftrightarrow \quad r_s \gtreqless 0 \, .$$

We conclude this section with a short remark regarding the question of feasibility of the underlying allocation. The long–run growth path of the economy is feasible, if (a) the expected growth rate is positive, (b) the propensity to consume is positive, and (c) expected lifetime utility is bounded. In the Pareto–efficient allocation, a positive propensity to consume is a sufficient condition to satisfy criterion (c) (see Merton, 1969), since lifetime utility can be determined as

$$V^*(0) = \frac{{\mu^*}^{-\rho} k(0)^{1-\rho}}{1-\rho} \, , \tag{28}$$

where in comparison to (19), also $\mu^* = \beta - (1-\rho)\left(\psi^* - \frac{1}{2}\rho A^2\sigma^2\right)$.

3 Distributive Disturbances

A modified consumer problem The analysis of the variances and covariances of the macroeconomic variables derived in the basic model of section 2 showed that the equilibrium relationships are perfectly correlated on the long–run growth path. As we already argued above, this outcome does not fit well to empirically observed movements over the business cycle. While we are not able to replicate a realistic lag structure (e. g. of labor incomes) in the context discussed here, it is nevertheless possible to incorporate the notion of an imperfect correlation between factor incomes and aggregate output by introducing distributive shocks as a second source of uncertainty. These then can be thought of as a compound of random factors disturbing the functional income distribution.

Our analysis is related to Femminis (1995) who discusses the possibility of adverse welfare effects emerging from risk sharing activities. He also examines distributive disturbances in a stochastic endogenous growth context, where, the lifetime present value of labor incomes (i. e. 'human wealth') is treated as a hedgeable asset. Contrary, our focus lies on the implications of distributive shocks for the growth process. Since these disturbances only have a long–lasting effect on individual optimization if they are related proportionally to the capital stock, we assume the following the stochastic processes for capital and wage incomes

$$\begin{aligned} \mathrm{d}P &= rk\,(\mathrm{d}t + \mathrm{d}z + \mathrm{d}v)\ , \\ \mathrm{d}W &= w(\mathrm{d}t + \mathrm{d}z) + \mathrm{d}u\ , \end{aligned} \tag{29}$$

where $\mathrm{d}z$ again denotes the technological diffusion, $\mathrm{d}v$ represents the exogenous distributive shock with zero mean and variance $\sigma_v^2\mathrm{d}t$, while the distributive wage process $\mathrm{d}u$ is to be determined endogenously in equilibrium.

The consumer maximizes lifetime utility (1) subject to the modified intertemporal budget constraint

$$\mathrm{d}k = [rk + w - c]\,\mathrm{d}t + (rk + w)\,\mathrm{d}z + rk\,\mathrm{d}v + \mathrm{d}u\ . \tag{30}$$

The variance of the capital stock changes to

$$\sigma_k^2 = (rk+w)^2\sigma_z^2 + (rk)^2\sigma_v^2 + \sigma_u^2 + [rk(rk+w)\,\sigma_{vz} + (rk+w)\,\sigma_{uz} + rk\sigma_{uv}]\ ,$$

where the instantaneous covariances are given by $\sigma_{xy} = \rho_{xy}\sigma_x\sigma_y$.

Macroeconomic equilibrium The distributive shock does not affect the market clearing values of factor prices which are determined according to (3) by the standard marginal productivity conditions. Furthermore, an equilibrium requires the redistribution between the two factors of production to be a zero–sum process, such that

$$\mathrm{d}u = -\alpha A k\, \mathrm{d}v \,, \tag{31}$$

and no resources are wasted. This implies for the equilibrium wage process that $\mathrm{d}W = AK[\gamma(\mathrm{d}t + \mathrm{d}z) - \alpha \mathrm{d}v]$. Obviously, the two diffusions $\mathrm{d}z, \mathrm{d}v$ are counteracting in the determination of the wage process, whereas they act jointly in the stochastic process of capital revenues, $\mathrm{d}P = \alpha AK(\mathrm{d}t + \mathrm{d}z + \mathrm{d}v)$.

Performing optimization according to (1) and the modified budget constraint (30), yields the following equilibrium values for the key macroeconomic relationships

$$\mu = \tfrac{1}{\rho}\left[\beta + \alpha A(\rho - 1)\right] + \gamma A + \tfrac{1}{2}A^2\sigma_z^2\left[\alpha(1-\rho) - \gamma(\rho+1)\right] + \alpha A^2\sigma_{vz} \,, \tag{32}$$

$$\psi = \tfrac{1}{\rho}(\alpha A - \beta) + \tfrac{1}{2}A^2\sigma_z^2\left[\alpha(\rho - 1) + \gamma(\rho+1)\right] - \alpha A^2\sigma_{vz} \,. \tag{33}$$

Compared to (14) and (17), both variables are augmented with the term $\alpha A^2\sigma_{vz}$ reflecting the consumer's response to the distributive shock. The sign of this additional component depends on the correlation between the productivity shock and the distributive disturbance, as measured by the instantaneous covariance $\sigma_{vz} = \rho_{vz}\sigma_z\sigma_v$. If the two sources of randomness are uncorrelated, that is $\rho_{vz} = 0$, the effects from a stochastic redistribution between factors exactly offset in equilibrium and the last term on the RHS of (32) and (33) vanishes. Empirical evidence suggests the correlation to be positive, thereby implying a higher probability of a positive realization of the productivity shock being accompanied by a distributive shock favoring capital incomes.

Since we excluded systematic components of redistribution from our analysis, reflecting, for instance, rule–based public tax–transfer schemes, the certainty equivalent parts of the propensity to consume (32) and the expected growth rate (33) remain unchanged. Given a positive sign of the covariance σ_{vz}, the presence of distributive shocks implies a higher propensity to consume out of capital and, naturally, lower expected growth. Due to the distributive disturbance, capital incomes have become riskier, which also immediately meets the eye, if we consider the certainty equivalent to capital return

$$r_s = \alpha A - \rho\alpha A^2\left(\sigma_z^2 + \sigma_{vz}\right) \,, \tag{34}$$

where the agents demand a higher associated risk premium, while the mean capital return remains unchanged. The increased riskiness of capital income gives rise to an additional substitution effect, inducing the household to raise present consumption and to reduce savings.

Variances and covariances in macroeconomic equilibrium The distributive disturbance does not affect aggregate production. For this reason the variance of output on the long–run growth path remains unchanged. The same argument applies to the variance of the aggregate capital stock and the stochastic growth rate, where the effects stemming from the distributive shock completely offset in equilibrium, due to its zero–sum nature. Changes can be observed for the correlation between factor incomes and between factor incomes and output. Due to the distributive shock, the income shares are stochastic, with expected values α and γ respectively. The distributive shock has a smoothing effect on the variability of wages, whereas, as already mentioned above, the riskiness of capital incomes increases. Moreover, if $\gamma\,\mathrm{d}z = \alpha\,\mathrm{d}v$ in (29), the randomness of labor incomes can be eliminated completely and capital incomes absorb the entire risk. This case corresponds to the assumptions Turnovsky (2000*b*) stated on the distribution of factor income risk for the analysis of elastic labor supply.

The standard deviations and covariances of the macroeconomic equilibrium can be summarized as follows:

$$\begin{aligned}
&\sigma_Y = \sigma_K = AK\sigma_z\,, \qquad \sigma_C = \mu\sigma_Y\,, \qquad \sigma_{\tilde{\psi}} = A\sigma\,,\\
&\sigma_W = AK\left(\gamma^2\sigma_z^2 + \alpha^2\sigma_v^2 - 2\alpha\gamma\sigma_{vz}\right)^{\frac{1}{2}}, \quad \sigma_P = \alpha AK\left(\sigma_z^2 + \sigma_v^2 + 2\sigma_{vz}\right)^{\frac{1}{2}}, \qquad (35)\\
&\rho_{YW} = \frac{\gamma\sigma_z - \alpha\sigma_{vz}/\sigma_z}{\left(\gamma^2\sigma_z^2 + \alpha^2\sigma_v^2 - 2\alpha\gamma\sigma_{vz}\right)^{\frac{1}{2}}}\,, \qquad \rho_{YP} = \frac{\sigma_z + \sigma_{vz}/\sigma_z}{\left(\sigma_z^2 + \sigma_v^2 + 2\sigma_{vz}\right)^{\frac{1}{2}}}\,,\\
&\rho_{PW} = \frac{\gamma\sigma_z^2 + (\gamma-\alpha)\sigma_{vz} - \alpha\sigma_v^2}{\left[\left(\sigma_z^2 + \sigma_v^2 + 2\sigma_{vz}\right)\left(\gamma^2\sigma_z^2 + \alpha^2\sigma_v^2 - 2\alpha\gamma\sigma_{vz}\right)\right]^{\frac{1}{2}}}
\end{aligned}$$

A comparison of (35) with (20) shows that in the presence of distributive disturbances, which are positively correlated with productivity shocks, the covariance between output and wage incomes is reduced, a result that is closer to empirical observations. The correlation may even vanish, if the effects from the two sources of uncertainty offset. Additionally, labor and capital incomes are no longer perfectly correlated, which also matches empirical findings. The variability of the intertemporal consumption path increases due to the rise in the equilibrium value of the expected propensity to consume (32), while the volatility of aggregate output remains unchanged.

4 Elastic Labor Supply

The model and intertemporal optimization The last part of this paper examines the question, to what extent income risks affect the equilibrium long–run expected growth path if we additionally allow labor supply to be elastic. We assume a momentary utility function in consumption $c(t)$ and leisure $l(t)$ of the form

$$U[c(t), l(t)] = \ln c(t) + \frac{l(t)^{1-\delta}}{1-\delta} \quad \text{if} \quad \delta > 0, \delta \neq 1 \,, \tag{36}$$

and $\ln c(t) + \ln l(t)$ if $\delta = 1$, where δ denotes the reciprocal of the intertemporal elasticity of substitution of leisure and can also be viewed as a coefficient of risk aversion, which implicitly measures the household's disliking of labor income risk.

Momentary utility (36) displays several important features: First, by recalling the arguments from the benchmark model, we expect the intertemporal income and substitution effects related to capital risk to completely offset, which simplifies the analysis considerably and allows us to focus on labor income risk. Second, due to the additively–separable structure of (36) the cross partial derivatives are zero, thereby eliminating effects from leisure on the marginal utility of consumption and *vice versa*. Third, as shown by King and Rebelo (1999), the consumer preferences (36) are consistent with a balanced growth path, where consumption growth is constant if capital returns are time–invariant, and the time fractions allocated to labor and leisure are constant too, if, again, the interest rate is constant and wages grow at a constant rate.

If labor supply is elastic and $(1-l)$ denotes labor input, the production technology available to the representative firm changes to:

$$\mathrm{d}y = Ak^{\alpha}(1-l)^{\gamma}K^{1-\alpha}\,(\mathrm{d}t + \mathrm{d}z) \ , \tag{37}$$

and modifies the intertemporal budget constraint $\mathrm{d}k = \mathrm{d}y - c\,\mathrm{d}t$ accordingly. The consumer's problem now is to select her rate of consumption and the fraction of time devoted to leisure activities in order to maximize lifetime utility according to (1) and (36) subject to her intertemporal budget constraint. The first–order conditions associated with this problem are:

$$\frac{\partial H}{\partial c} = U_c(c,l)\, e^{-\beta t} - \lambda = 0\,, \tag{38a}$$

$$\frac{\partial H}{\partial l} = U_l(c,l)\, e^{-\beta t} - \lambda w + \frac{1}{2}\frac{\partial \lambda}{\partial k}\frac{\partial \sigma_k^2}{\partial l} = 0\,, \tag{38b}$$

$$\mathrm{d}\lambda = -\left[\lambda r + \frac{1}{2}\frac{\partial \lambda}{\partial k}\frac{\partial \sigma_k^2}{\partial k}\right] \mathrm{d}t + \frac{\partial \lambda}{\partial k}\left[rk + w(1-l)\right] \mathrm{d}z\,. \tag{38c}$$

The new condition (38b) relates the marginal utility of leisure to the shadow price λ, but additionally accounts for the random nature of labor incomes. The first–order conditions related to the optimal choice of consumption and leisure, (38a) and (38b), imply the following familiar relationship:

$$\frac{U_l}{U_c} = w\left[1 - \sigma^2(r + (1-l)w/k)\right]\,. \tag{39}$$

Expression (39) is a modified version of the condition usually derived for endogenous labor–leisure choice, which presumes the marginal rate of substitution between leisure and consumption to be equal to the opportunity costs of leisure, as measured by the real wage rate. In the stochastic context considered here, the marginal rate of substitution is equated to the risk–adjusted expected wage rate, given by the RHS of (39), which is smaller than its deterministic counterpart.

Macroeconomic equilibrium The solution of the model now follows the procedure already described in section 2. For the sake of analytical brevity, we focus on the conditions characterizing a macroeconomic equilibrium and on comparative static results. Besides a time–invariant propensity to consume, the solution conjecture now also involves that the time–fractions allotted to labor and leisure be constant on the long–run growth path. The equilibrium expected wage rate and the expected rate of capital return are given by:

$$w = A\gamma(1-l)^{\gamma-1}k\,, \qquad r = \alpha A(1-l)^{\gamma}\,, \tag{40}$$

where, as before, the wage rate grows linearly in the capital stock, while the real interest rate remains constant on the steady–state growth path. Furthermore, the wage rate decreases with an increase in labor supply $1-l$, whereas capital productivity and therefore the expected capital return is raised.

Differentiating the first–order condition (38a) with respect to time and equating it with (38c), while taking account of the solution conjecture $\mu = c/k$, condition (11), and (40), this finally implies an expression for the propensity to consume, characterizing the consumption–savings tradeoff. Similarly, by utilizing (11) and substituting the first and higher–order derivatives of momentary utility (36) into condition (38b), this, too, can be solved for an expression of the propensity to consume, this time reflecting the consumption–leisure tradeoff. Let μ_1 represent the first and μ_2 denote the latter, they can be determined as

$$\mu_1 = \beta + A\gamma(1-l)^\gamma \left[1 - A\sigma^2(1-l)^\gamma\right] , \tag{41a}$$

$$\mu_2 = A\gamma l^\delta (1-l)^{\gamma-1} \left[1 - A\sigma^2(1-l)^\gamma\right] . \tag{41b}$$

Both expressions are non–linear functions in leisure. If we now compare the corresponding condition for μ from the inelastic labor supply setting (eq. (14)), with (41a), it becomes obvious that the first can be obtained from the latter, if we set $\rho = 1$ and $l = 0$, which also shows that the effects from capital risk cancel out for logarithmic preferences. Regarding the influence of factor income risk on the household's consumption decision, we find that only the response to labor income risk remains effective.

Analogously to the propensity to consume, we obtain two expressions for the expected growth rate of the economy from the consumer's budget constraint $\mathrm{d}k = [rk + (1-l)w - c]\,\mathrm{d}t + [rk + (1-l)w]\,\mathrm{d}z$ and the market clearing condition $\mathrm{d}K = \mathrm{d}Y - C\,\mathrm{d}t$. Denoting these with ψ_1 and ψ_2, they can be derived as

$$\psi_1 = \alpha A(1-l)^\gamma - \beta + \gamma A^2\sigma^2(1-l)^{2\gamma} , \tag{42a}$$

$$\psi_2 = A(1-l)^\gamma \left[1 - \tfrac{\gamma l^\delta}{1-l}\left[1 - A\sigma^2(1-l)^\gamma\right]\right] , \tag{42b}$$

and similar to (41a) and (41b) are non–linear functions in leisure. The two conditions describe tradeoff loci between expected growth and leisure. The equilibrium value of the certainty equivalent to capital return can be derived from (38c) and is given by

$$r_s = \alpha A(1-l)^\gamma \left[1 - A\sigma^2(1-l)^\gamma\right] . \tag{43}$$

As already mentioned above, a balanced growth path also requires the fractions of time devoted to labor and leisure to be constant in equilibrium. Due to the non–linear nature of (42a) and (42b) it is not possible to solve explicitly for the equilibrium value of leisure. Together with the associated equilibrium expected growth rate, it is determined implicitly by the intersection of ψ_1 with ψ_2. If we define for analytical convenience $\Psi[\psi_1(l), \psi_2(l)] \equiv \psi_2 - \psi_1$, the condition for steady–state growth is given by

$$\Psi[\psi_1(l), \psi_2(l)] = \beta + A\gamma(1-l)^\gamma \left(1 - \tfrac{l^\delta}{1-l}\right)\left[1 - A\sigma^2(1-l)^\gamma\right] = 0 . \tag{44}$$

Before we proceed with exploring the question of existence of a balanced growth path in more detail, we want to give a short summary of the associated conditions characterizing an efficient allocation.

The efficient allocation The benevolent social planner takes account of the knowledge spillovers in performing optimization and rewards capital the higher social return. Under the preference specification assumed with (36) we obtain the following pairs of propensities to consume

$$\mu_1^* = \beta\,, \tag{45a}$$

$$\mu_2^* = \mu_2 = A\gamma l^{\delta}(1-l)^{\gamma-1}\left[1 - A\sigma^2(1-l)^{\gamma}\right]\,, \tag{45b}$$

and expected growth rates

$$\psi_1^* = A(1-l)^{\gamma} - \beta\,, \tag{45c}$$

$$\psi_2^* = \psi_2 = A(1-l)^{\gamma}\left[1 - \tfrac{\gamma l^{\delta}}{1-l}\left[1 - A\sigma^2(1-l)^{\gamma}\right]\right]\,. \tag{45d}$$

Conditions (45a) and (45c) display the well–known certainty equivalent result which is typical for logarithmic preferences in the presence of a pure capital risk. The propensity to consume is determined entirely by the rate of time preference, being positive by assumption. The expected growth rate, (45c), also is independent of the underlying risk. Conditions (45b) and (45d), stemming from the consumption–leisure tradeoff, are unaffected by the human capital externality and therefore coincide with the corresponding conditions derived for the competitive economy, (41b) and (42b). Additionally, the certainty equivalent to capital return equals the value obtained in (43) with α set to unity.

We define $\Psi^* \equiv \psi_2^* - \psi_1^*$. Then, the efficient fraction of time devoted to leisure is implicitly expressed in the following condition:

$$\Psi^*[\psi_1^*(l), \psi_2^*(l)] = \beta - A\gamma l^{\delta}(1-l)^{\gamma-1}\left[1 - A\sigma^2(1-l)^{\gamma}\right] = 0\,. \tag{46}$$

Existence of unique and multiple equilibria Regarding uniqueness of an equilibrium in the competitive economy as well as in the social optimum, we can state:

Definition 1 *A unique balanced growth path exists, if the two functions $\Psi[\psi_1(l), \psi_2(l)]$ and $\Psi^*[\psi_1^*(l), \psi_2^*(l)]$ satisfy the following conditions:*

(i) Ψ *and* Ψ^* *are continuous functions in the domain* $l \in (0,1)$.

(ii) Ψ *and* Ψ^* *are monotonic functions in the domain* $l \in (0,1)$.

(iii) The limits of Ψ and Ψ^* respectively are of opposite sign, that is

$$\operatorname{sgn} \lim_{l\to 0} \Psi = -\operatorname{sgn} \lim_{l\to 1} \Psi \quad \textit{and} \quad \operatorname{sgn} \lim_{l\to 0} \Psi^* = -\operatorname{sgn} \lim_{l\to 1} \Psi^* \ .$$

If we now look at the definitions (44) and (46) of Ψ and Ψ^*, it becomes obvious that the question of existence of a balanced growth path is closely related to the sign of the certainty equivalent to capital return. While condition (44) does not exclude the possibility that an equilibrium with $r_s < 0$ exists, this is not true for Ψ^*. In general, we can state:

Proposition 2 (Condition for a unique equilibrium) *The macroeconomic equilibrium of the competitive economy is consistent with a unique balanced growth path, if the certainty equivalent to capital return is positive. A positive value of r_s is necessary and sufficient for existence and uniqueness of steady–state growth in the Pareto–efficient allocation. There, the case of $r_s = 0$ is not consistent with balanced growth.*

Proof: The equilibrium conditions Ψ and Ψ^* are continuous and monotonically decreasing in the interval $l \in (0,1)$ for $r_s > 0$ (see Appendix A). The limits for $l \to 0$ and $l \to 1$ are given by:

$$\lim_{l\to 0} \Psi = \beta + A\gamma\left(1 - A\sigma^2\right) , \qquad \lim_{l\to 0} \Psi^* = \beta \, , \qquad \lim_{l\to 1} \Psi = \lim_{l\to 1} \Psi^* = -\infty \, ,$$

from which follows immediately that the third condition of Definition 1, requiring a change of signs, is satisfied for $r_s > 0$, and Ψ as well as Ψ^* uniquely intersect with the horizontal axis. □

So, what about multiple equilibria in the competitive economy? If we return to the definition of the tradeoff locus Ψ from (44), we see that, hypothetically, balanced growth can also be consistent with a negative value of r_s, as long as the function $1 - l^{\delta}/(1-l)$ is positive at the equilibrium value of l. This expression is negatively sloped in l, attains its largest value in the limit $l \to 0$, and asymptotically tends towards minus infinity for $l \to 1$. Contrary, the expression $1 - A\sigma^2(1-l)^{\gamma}$ deciding upon the sign of r_s, attains its smallest value for $l \to 0$, while monotonically increasing towards its upper limit of unity for $l \to 1$. Consequently, if this function is sufficiently negative at the lower limit $l \to 0$, this altogether implies a hump–shaped curvature of Ψ, finally giving rise to the possibility of multiple equilibria.

Proposition 3 (Existence of multiple equilibria) *The tradeoff locus Ψ of the competitive economy possesses two roots in leisure l, if the certainty equivalent to capital return is sufficiently negative at the limit $l \to 0$, that is, if $\Psi(0) = \beta + A\gamma(1 - A\sigma^2) < 0$. The equilibrium associated with the lower value of l is characterized by dynamic inefficiency, where the expected growth rate* *exceeds the Pareto–optimal one. The second equilibrium, associated with the larger root, implies suboptimally low growth.*

Proof: by Proposition 1 and (A.4) to (A.6) in Appendix A.

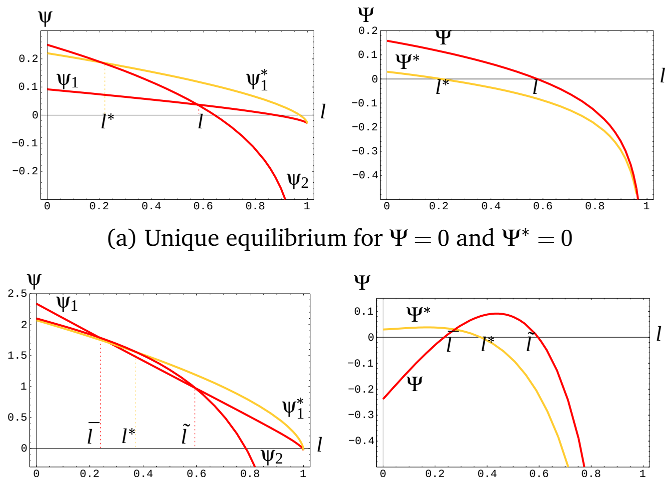
The results of Propositions 2 and 3 are displayed in figure 1.[7] The LHS of subfigure 1(a) focuses on the functions ψ_1 (ψ_1^* for the social optimum) and ψ_2, the first implicitly representing the tradeoff between consumption and savings, the latter reflecting the consumption–leisure tradeoff, whereas the RHS of subfigure 1(a) depicts the Ψ–locus for the competitive economy and the Ψ^*–locus for the social optimum. The intersections of the Ψ, Ψ^*–graphs with the horizontal axis determine the fraction of leisure time being consistent with balanced growth. The macroeconomic equilibrium of the competitive economy is characterized by a larger fraction of time devoted to leisure and consequently by lower expected growth compared to the efficient allocation.

Subfigure 1(b) shows the case of multiple equilibria in the competitive economy and contrasts these with the unique balanced growth path of the efficient economy. We observe that ψ_1 and ψ_2 intersect twice, which is displayed on the LHS of subfigure 1(b). This scenario corresponds to the two roots of Ψ on the RHS of subfigure 1(b), implying the two equilibrium values $\bar{l}$ and $\tilde{l}$. The lower value, $\bar{l}$, corresponds to dynamic inefficiency, yielding a suboptimally high expected growth rate compared to the efficient allocation, whereas the larger value, $\tilde{l}$, is associated with suboptimally low expected growth. In $\bar{l}$, households work too much and enjoy too little leisure, while the opposite holds in $\tilde{l}$. Nevertheless, we are able to exclude the dynamically inefficient equilibrium from the set of feasible allocations. By utilizing (39) and (41b) we find:[8]

Proposition 4 *The equilibrium associated with a negative certainty equivalent to capital return implies a negative equilibrium value of the propensity to consume. Therefore, it is not consistent with a feasible allocation.*

[7]The parameters were set according to $\alpha = 0.35, \beta = 0.03, \gamma = 0.65, \delta = 1.0, \sigma = 0.06$, which corresponds to empirical estimates; see also Ramey and Ramey (1995) and Turnovsky (2000*b*). King and Rebelo (1999) use a momentary utility function of the form $U(c,l) = \ln c + \vartheta l^{1-\delta}/(1-\delta)$ and adjust the parameter ϑ correspondingly to match empirical findings of $1-l = 0.2$. In Subfigure 1(b), the variance σ^2 was set arbitrarily in order to illustrate the results of Proposition 3.

[8]The statement of Proposition 4 does not interfere with the feasibility of a dynamically inefficient allocation in the model with inelastic labor supply. There, a negative value of r_s does not necessarily imply a negative value of the corresponding propensity to consume.



(a) Unique equilibrium for $\Psi = 0$ and $\Psi^* = 0$

(b) Multiple equilibria for $\Psi = 0$

Figure 1: *Equilibrium with elastic labor supply*

So what remains is the equilibrium related to suboptimally low growth. In comparison to the model with inelastic labor supply, the household now has the option to avoid uncertain labor incomes and the associated consumption risk by extending her demand for riskless leisure; see Bodie *et al.* (1992). So, besides the fact that agents continue to underestimate the true (social) return to capital as they neglect the technological spillovers, the real interest rate is also affected by the endogenously determined amount of labor. The aggregate labor supply increases with a rising wage rate, which is tantamount to higher opportunity costs of consumption related to leisure. The larger the fraction of time devoted to labor, the higher capital productivity. This raises the marginal utility of future consumption. In order to balance marginal utilities over time, the rate of return to current consumption has to rise too, inducing the households to consume less and increase savings. In general, the efficient allocation is characterized by larger rates of return to consumption and leisure, whereas in the competitive economy the households work less and consume a larger fraction of their permanent income, which altogether implies a reduction in the expected balanced growth rate of the economy.

Comparative statics This section concludes with a brief analysis of how changes in the primitives of the model affect the equilibrium amount of labor supply as implicitly described by the equilibrium condition $\Psi = 0$, given by (44). We are primarily interested in two factors: (a) the variance of the productivity shock, σ^2, providing us with an intuition of how increases or decreases in risk affect the equilibrium labor–leisure choice, and (b) the elasticity of marginal utility of leisure, δ, which measures the extent of the household's willingness to substitute intertemporally, or her degree of risk aversion towards wage income risk, respectively. By the implicit–function theorem, we obtain

$$\frac{\mathrm{d}l}{\mathrm{d}\sigma^2} = \frac{A\gamma(1-l)^{1+\gamma}\left(\frac{l^\delta}{1-l}-1\right)}{\gamma[1-2A\sigma^2(1-l)^\gamma]+l^\delta[1-A\sigma^2(1-l)^\gamma]\left(\frac{\alpha}{1-l}+\frac{\delta}{l}\right)+A\gamma l^\delta\sigma^2(1-l)^{\gamma-1}}, \tag{47}$$

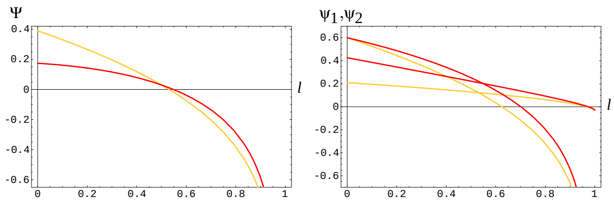
which is certainly positive for $1 > 2A\sigma^2(1-l)^\gamma$, and

$$\frac{\mathrm{d}l}{\mathrm{d}\delta} = -\frac{l^\delta \ln l}{\gamma+l^\delta\left(\frac{\alpha}{1-l}+\frac{\delta}{l}\right)-[A\gamma\sigma^2(1-l)^\gamma][1-A\sigma^2(1-l)^\gamma]^{-1}\left(1-\frac{l^\delta}{1-l}\right)} > 0, \tag{48}$$

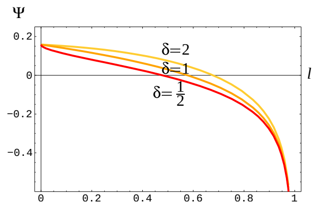
which is definitely positive, if we take account of the conditions, that in equilibrium $r_s > 0$ and consequently $1 < l^\delta/(1-l)$, and that $l \in (0,1)$, implying $\ln l < 0$.

The household responds to a rise in risk in a manner similar to an increase in the degree of risk aversion, namely, with a reduction in labor supply. While in the first case the tradeoff locus Ψ is turned anti–clockwise, it is bent upwards in the latter. Both results are illustrated in Figure 2. Subfigure 2(a) compares the tradeoff loci of the deterministic economy ($\sigma = 0$) with the stochastic one. An equilibrium in the stochastically growing economy is characterized by a larger demand for leisure. That this does not necessarily imply a smaller growth rate, is displayed on the RHS of subfigure 2(a). As we have shown in the previous sections, the consumer saves out of precautionary motives in the presence of labor income risk. These savings increase with a rise in the variance of the technological shock, and so does the expected growth rate ψ_1 as given by (42a), whereas the propensity to consume (41a) decreases. The household switches away from current consumption and tries to compensate the utility loss with a larger demand for leisure.

A rise in δ reflects that the consumer is less willing to substitute leisure over time and prefers a uniform pattern regarding the division of available time on labor and leisure. The agent is willing to forego with current and future consumption in order to satisfy her rising demand for leisure. This result is illustrated in subfigure 2(b). The expected growth rate as well as the equilibrium value of the propensity to consume decline, which intuitively



(a) Changes in technological risk, σ



(b) Changes in δ

Figure 2: *Comparative static shifts in the equilibrium labor–leisure choice*

becomes clear by (41a) to (42b). Whereas μ_1 and ψ_1 (eqs. (41b) and (42b)) remain unaffected by changes of δ, the slope of μ_2 and ψ_2 becomes flatter. From this follows that the tradeoff loci intersect at larger values of l as δ rises. Since ψ_1 and μ_1 are monotonically decreasing in l, this finally implies lower equilibrium values for both, the propensity to consume and the expected growth rate.

5 Conclusion

This paper has explored several dimensions of how capital and labor income risk impact on the long–run expected growth path of the economy. We examined inelastic as well as elastic labor supply and also addressed the issue of distributive disturbances. We employed the Romer (1986) endogenous growth model with human capital externalities as general framework, which was extended with productivity shocks. The analysis of the benchmark model of inelastic labor supply revealed that the major results derived by Sandmo (1970) for CRRA–preferences are preserved to the extent that labor income risk always induces precautionary savings, thereby enhancing growth, whereas the consumer's response to capital risk is governed by the well–known counter–acting intertemporal income and substitution effects giving rise to associated ambiguous growth effects. We have shown that the presence of labor income risk and related precautionary savings contribute to a better explanation of the relative smoothness of aggregate consumption in the data (cf. Caballero, 1990). Regarding the Pareto–inferior nature of the competitive equilibrium, we found that the agents, by focusing only on private factor incomes, not only underestimate the mean return to capital, but do so also for the volatility of future capital income. This leads to a situation where risk–induced accumulation is too large, thereby implying the possible emergence of excessively high growth. We were able to relate the outcomes to the size of the risk premium, which the risk averse households demand for bearing the investment risk.

The productivity shock being the single source of uncertainty of the economy in the benchmark model, the factor returns as well as consumption are perfectly correlated with aggregate output. This reflects a standard outcome of the C–CAPM, but also gives a notion of why the model performs poorly when confronted with time series data. For this reason we also studied the impact of distributive disturbances on the expected balanced growth path of the economy and on the observed correlations between the macroeconomic variables. We find, that the increase in the volatility of capital returns due the distributive shock gives rise to an additional substitution effect, which eventually reduces expected growth. Regarding the correlations between factor incomes, we can state that the increase in the volatility of capital returns naturally is accompanied by less volatile labor incomes. While capital incomes remain positively correlated with output, the correlation between labor incomes and output is weakened and, depending on the size of the covariance between the two shocks, may ultimately also become negative.

The last section of this paper was devoted to the analysis of elastic labor supply. By allowing for an endogenous labor–leisure choice, the risk averse consumer has an additional choice variable to avoid the riskiness of income flows, namely by increasing her demand for leisure. Compared to the deterministic economy, this leads to a smaller labor supply but not necessarily to a smaller growth rate, since the general desire to save out of precautionary motives is left unaffected by extending momentary utility with preferences for leisure. We derived conditions related to the existence of a unique equilibrium but also found conditions under which two equilibria exist in the competitive economy, one being associated with suboptimally low labor supply and growth compared to the efficient allocation, the other combining inefficiently large labor inputs with suboptimally high growth. As before it is possible to relate the outcomes to the size of the risk premium, but contrary to the model with inelastic labor supply, we were able to rule out the dynamically inefficient equilibrium for feasibility reasons.

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A Elastic Labor Supply

Continuity and Monotonicity of Ψ and Ψ^* Recall that $\psi_2 = \psi_2^*$. The functions Ψ and Ψ^* are continuous, if they are differentiable in l in the entire domain $l \in (0,1)$, which is satisfied here

$$\frac{\partial \psi_1}{\partial l} = -\gamma A(1-l)^{\gamma-1}\left[\alpha + 2\gamma A\sigma^2(1-l)^{\gamma}\right] < 0\,, \tag{A.1}$$

$$\frac{\partial \psi_1^*}{\partial l} = -\gamma A(1-l)^{\gamma-1} < 0\,, \tag{A.2}$$

$$\frac{\partial \psi_2}{\partial l} = -\frac{\gamma}{1-l}\left[\psi_2 + A\gamma l^{\delta}(1-l)^{\gamma}\left(\left(1-A\sigma^2(1-l)^{\gamma}\right)\left(\frac{\delta}{l}+\frac{1}{1-l}\right)+\frac{\gamma A\sigma^2}{(1-l)^{1-\gamma}}\right)\right] < 0 \quad \text{if} \quad r_s > 0\,. \tag{A.3}$$

The tradeoff loci ψ_1, ψ_2, ψ_1^* are monotonically decreasing in l and so are $\Psi = \psi_2 - \psi_1$ and $\Psi^* = \psi_2^* - \psi_1^*$.

Existence of multiple equilibria Define the following functions $f(l) \equiv 1 - \frac{l^{\delta}}{1-l}$ and $g(l) \equiv 1 - A\sigma^2(1-l)^{\gamma}$. Then

$$f'(l) < 0\,, \qquad \lim_{l\to 0} f(l) = 1\,, \qquad \lim_{l\to 1} f(l) = -\infty\,, \tag{A.4}$$

$$g'(l) > 0\,, \qquad \lim_{l\to 0} g(l) = 1 - A\sigma^2\,, \qquad \lim_{l\to 1} g(l) = 1\,. \tag{A.5}$$

$f(l)$ changes signs only once in the domain $l \in (0,1)$. The same result can be obtained for $g(l)$ if $g(0) < 0$. This altogether implies a hump–shaped curvature of the function

$$\Phi \equiv \frac{\beta}{A\gamma(1-l)^{\gamma}} + f(l)\,g(l)\,, \tag{A.6}$$

which can be obtained upon rearranging Ψ from (44). In order to prove the existence of two intersection points associated with values of l satisfying the equilibrium condition $\Psi = 0$, we first have to exclude the possibility that the function Φ, is negative in the entire domain $l \in (0,1)$ and hence never satisfies the condition $\Phi = 0$. In short, what we are looking for is the smallest maximum of Φ, where, in general, a maximum of Φ requires $f(l)$ and $g(l)$ to be of identical sign. With $g(l)$ starting in the negative quadrant, while $f(l)$ starts in the positive one, the smallest maximum of the product $f(l)g(l)$ is attained, when both functions change signs for an identical value of l. But, in this case $f(l) = g(l) = 0$, and the smallest maximum of Φ is given by the first term in definition (A.6), that is by $\beta/(A\gamma(1-l)^{\gamma}) > 0$. This excludes the possibility of Φ being entirely located in the negative quadrant.

The function Φ is monotonically decreasing to the LHS and the RHS of its maximum value. From this follows immediately that Φ has two intersection points with the horizontal axis, corresponding to two equilibrium values of l satisfying the condition $\Psi = 0$. By (A.4) and (A.5), the smaller one is associated with $f(l) > 0$ and $g(l) < 0$, while the larger one corresponds to $f(l) < 0$ and $g(l) > 0$.