RATIONAL BUBBLES AND CHANGING DEGREE OF FRACTIONAL INTEGRATION

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Abstract

In this article we provide evidence for a rational bubble in S&P 500 stock prices by applying a test for changing persistence under fractional integration proposed by Sibbertsen and Kruse (2007). We find strong evidence for stationary long memory before the estimated change point in 1955 and a unit root afterwards. These results bring two empirical findings in line: on one hand they confirm the previous result of fractional integration and on the other hand they support the hypothesis of a rational bubble.

JEL-number: C12, C22, G12.

Keywords: Fractional integration; bubbles; changing persistence

1 Introduction

In this article we provide evidence for a rational bubble in S&P 500 stock prices by applying a test for changing persistence under fractional integration. Koustas and Serletis (2005) find strong evidence for the existence of long memory in the S&P 500 log dividend yield and their results support the hypothesis of no rational bubble. However, the authors did not take account for a potential change in the fractional degree of integration. We apply a suitable test proposed by Sibbertsen and Kruse (2007) and find a significant break in the memory of the S&P 500 log dividend yield that is located at November, 1955. This breakpoint was also found by Sollis (2006) who applied tests for a change in persistence that are designed for the $I(0)/I(1)$-framework. We find strong evidence for stationary long memory before the break in 1955 and a unit root afterwards. These results confirm on one hand the previous result of fractional integration in this time series and on the other hand they are in line with other empirical studies that found evidence for a rational bubble in it.

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2 Testing for changing memory

We assume that the data generating process follows an $I(d)$ process as proposed by Granger and Joyeux (1980):

$$(1-L)^d y_t = \varepsilon_t,$$

where ε_t are i.i.d. random variables with mean zero and variance σ^2 and L denotes the lag operator ($L^k y_t \equiv y_{t-k}$). This process is said to be fractionally integrated of order d. The test proposed by Sibbertsen and Kruse (2007) considers the following pair of hypotheses,

$$\begin{aligned} H_0 &: \; d = d_0 \;\text{ for all } t \\ H_1 &: \; d = d_1 \;\text{ for }\; t = 1, \ldots, [\tau T] \\ & \;\;\; d = d_2 \;\text{ for }\; t = [\tau T]+1, \ldots, T \end{aligned}$$

where $[x]$ denotes the biggest integer smaller than x. The differencing parameter is restricted to $0 \leq d_0 < 3/2$ under H_0, while $0 \leq d_1 < 1/2$ and $1/2 < d_2 < 3/2$. Note that, d_1 and d_2 can be interchanged. This means that we test the null hypothesis of constant memory against a change from stationary ($0 \leq d_1 < 1/2$) to non-stationary ($1/2 < d_2 < 3/2$) long memory at $[\tau T]$ and vice versa. The test statistic is given by

$$R = \frac{\inf_{\tau \in \Lambda} K^f(\tau)}{\inf_{\tau \in \Lambda} K^r(\tau)},$$

where $K^f(\tau)$ and $K^r(\tau)$ are CUSUM of squared based statistics based on the forward and reversed residuals of the data generating process as given below. The relative breakpoint $\tau \in \Lambda \subset (0,1)$ is assumed to be unknown and a simple estimator is given below. In detail, the forward and reverse CUSUM of squared based statistics are defined by

$$K^f(\tau) = [\tau T]^{-2} \sum_{t=1}^{[\tau T]} \hat{v}_{t,\tau}^2$$

and

$$K^r(\tau) = (T - [\tau T])^{-2} \sum_{t=1}^{T-[\tau T]} \tilde{v}_{t,\tau}^2.$$

Here, $\hat{v}_{t,\tau}$ are the residuals from the OLS regression of y_t on a constant based on the observations up to $[\tau T]$. This is

$$\hat{v}_{t,\tau} = y_t - \overline{y}(\tau)$$

with $\overline{y}(\tau) = [\tau T]^{-1}\sum_{t=1}^{[\tau T]} y_t$. Similarly $\tilde{v}_{t,\tau}$ is defined for the reversed series $z_t \equiv y_{T-t+1}$. Thus, it is given by

$$\tilde{v}_{t,\tau} = z_t - \overline{z}(1-\tau)$$

with $\overline{z}(1-\tau) = (T - [\tau T])^{-1}\sum_{t=1}^{T-[\tau T]} z_t$. Since the limiting distribution of R depends on the memory parameter under the null hypothesis d_0, Sibbertsen and Kruse (2007) provide response curves that allow an easy computation of relevant critical values. Note that, when testing against a change from stationary to non-stationary memory the left tail of the distribution is relevant and vice versa. Furthermore, the authors prove consistency of the simple breakpoint estimator that is given by

$$\hat{\tau} = \inf_{\tau \in \Lambda} K^f(\tau).$$

3 Empirical evidence

The used monthly data set can be downloaded from Robert Shiller's web site[1]. The sample spans from January, 1871 to December, 2007 implying 1644 observations. The time series is depicted in Figure 1. The graph shows a clear change in the behaviour in the last third of the sample.

In a first step of our analysis, we estimate the long memory parameter by applying the log periodogram regression method proposed by Geweke and Porter-Hudak (1983). This estimator is based on an approximation of the spectral density near the origin. A crucial issue is the choice of number of frequencies m that are used to perform the log periodogram regression. Hurvich, Deo and Brodsky (1998) show that $m = o(T^{4/5})$ is MSE-optimal. On the other hand Geweke and Porter-Hudak suggest to use $m = o(T^{1/2})$ which means that higher frequencies are disregarded which implies that the estimator is less efficient. On the other hand, if the true DGP contains short-term dependencies which are usually represented by an ARMA(p,q) process, the GPH estimator based on $m = o(T^{1/2})$ is less biased. Henceforth, there is a tradeoff between bias and efficiency.

Davidson and Sibbertsen (2007) recently proposed a Hausman-type test for the bias in log-periodogram regressions that compares two GPH estimators using a different number of frequencies. Under the null hypothesis short-term dependencies are negligible and therefore a higher number of frequencies, $m = o(T^{4/5})$, can be used without running the risk of a bias. Under the alternative the authors suggest to use a lower number of frequencies, $m = o(T^{1/2})$. An application of this test leads to a rejection at the nominal five percent level of significance (p-value= 0.030).

[1] http://cowles.econ.yale.edu/faculty/shiller.htm/

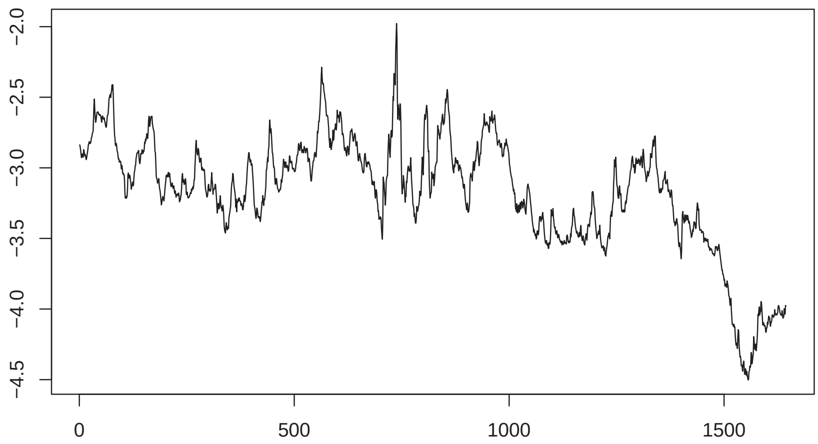

Figure 1: S&P 500 log dividend yield (January, 1871 to December, 2007).

The estimate of d_0 using $m = T^{1/2}$ is 0.82 which indicates a non-stationary long memory time series. Alternatively to the log periodogram regression approach we estimate an ARFI-MA$(p,d,0)$ model in order to account explicitly for short-term correlations represented by a finite AR component. MA components are omitted for simplicity. The model can be written as

$$(1-\alpha_1 L-\alpha_2 L^2-\ldots-\alpha_p L^p)(1-L)^d y_t = \mu+\varepsilon_t ,$$

where α_i denote the AR-parameters and μ is a constant. All parameters of this model are estimated jointly via nonlinear least squares which is often referred to as the conditional sum-of-squares (CSS) estimator that has been suggested by Beran (1995) and further studied in Chung and Baillie (1993) and Doornik and Ooms (1999). In contrast to exact maximum likelihood or modified profile likelihood estimation, the NLS estimator is also applicable for non-stationary ARFIMA models $(0.5 < d \leq 1)$. The optimal autoregressive lag length is chosen via AIC with a lower bound of zero and an upper bound of $p_4 = [4(T/100)^{1/4}] = 7$, cf. Schwert (1989).

Detailed NLS estimation results can be found in Table 1. Compared to the GPH estimate we obtain a slightly lower value of 0.61. A simple t-test of the null hypothesis of short memory $H_0 : d_0 = 0$ has to be strongly rejected. The Ljung-Box

Table 1: NLS estimation results for ARFIMA(p,d,0) models

	$T=1-1644$		$T=1-1019$		$T=1020-1644$	
μ	-3.079	(0.000)	-2.949	(0.000)		
d	0.612	(0.000)	0.327	(0.006)	0.949	(0.000)
α_1	0.712	(0.000)	1.011	(0.000)	0.324	(0.003)
α_2	-0.071	(0.162)	-0.191	(0.099)	-0.057	(0.227)
α_3	0.012	(0.787)	-0.021	(0.754)	0.050	(0.343)
α_4	0.061	(0.314)	0.114	(0.068)	-0.011	(0.806)
α_5	0.080	(0.048)			0.122	(0.015)
α_6					-0.086	(0.057)
$Q(12)$	4.717	(0.967)	7.769	(0.803)	2.938	(0.996)

Notes: P-values are reported in brackets beside the corresponding estimate or test statistic.

statistic Q with 12 lags is not significant which suggests that there is no remaining autocorrelation up to lag 12 left in the residuals.

The interval of potential breakpoints is set $\Lambda = [0.2, 0.8]$ which is a common choice in the literature. The test statistic R equals 0.35. Based on $\hat{d}_0 = 0.82$, critical values that are computed via response curves equal 0.47, 0.38 and 0.24 for the nominal 10, 5 and 1 percent level of significance, respectively. We have to reject the null hypothesis of constant memory in favor of the alternative that the memory increases for small values of R. Henceforth, we find evidence for changing memory at the five percent level. When using the ARFIMA model based estimate of d_0 the critical values are 0.70, 0.61 and 0.48, respectively. Thus, H_0 has to be rejected even at the one percent level of significance. We therefore conclude, that there might be a change in d from d_1 to d_2.

The estimated breakpoint is at observation 1019 which corresponds to November, 1955. A very similar breakpoint was found by Sollis (2006) who applied the Leybourne et al. (2003) test for a unit root against a change from $I(0)$ to $I(1)$. The GPH estimate of the memory parameter before the break (based on $T_1 = 1019$ observations) is $\hat{d}_1 = 0.37$ and significantly different from zero (p-value$= 0.000$). This result suggests that the S&P 500 log dividend yield is fractionally integrated before November, 1955. Considering the estimated ARFIMA model for the first subsample, we find further evidence for long-range dependence since the estimate $\hat{d}_1 = 0.327$ is highly significant (p-value$= 0.006$). After the break the GPH estimate increases to $\hat{d}_2 = 1.09$ which is close to unity but higher than one suggesting a potential unit root. This estimate is based on $T_2 = 625$ observations. Again, the ARFIMA model based estimate ($\hat{d}_2 = 0.949$) is lower but even closer to unity.

In order to carry out a formal and suitable test of the unit root hypothesis against long memory we apply the fractional Dickey-Fuller test proposed by Dolado et al. (2002). Their procedure is based on the test regression

$$\Delta^{\delta_0} y_t = \phi \Delta^{\delta_1} y_{t-1} + \sum_{i=1}^{p} \lambda_i \Delta y_{t-i} + \varepsilon_t$$

where $\delta_0 = 1$ in our application, δ_1 is unknown and has to be estimated from the data. Note, that the estimator for δ_1 has to be $T^{1/2}$-consistent, we therefore employ the parametric NLS estimator proposed by Beran (1995). Regarding the test regression the relevant pair of hypotheses is $H_0 : \phi = 0$ versus $H_1 : \phi < 0$. Dolado et al. (2002) prove that the limiting distribution of the t-statistic for H_0 is standard normal if $0.5 \leq \delta_1 < 1$ which is the relevant case in our application. For further details, the reader is referred to Dolado et. al (2002). As before, the maximum lag length is set equal to $p_4 = [4(T_2/100)^{1/4}] = 6$. The optimal length is chosen with the Schwarz information criteria and equals zero.The estimated test regression without lags of $\triangle y_t$ is given by

$$\Delta y_t = 0.245 \Delta^{0.949} y_{t-1} + \hat{\varepsilon}_t \ .$$

Since the test statistic $t_\phi = 6.593$ is not significant at conventional levels (p-value= 1), we are not able to reject the null hypothesis of a unit root in the second subsample which hints at a rationale bubble because the no-bubbles restriction is not fulfilled in this case.

4 Conclusions

This paper provides evidence that the time series properties of the S&P 500 log dividend yield are changing over time. We found by applying recent tests that the time series is stationary fractionally integrated before November, 1955 and exhibits a unit root afterwards. The presence of a unit root in the second subsample suggests a rationale bubble in the S&P 500 stock price.

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