Do algebraic numbers follow Khinchin's Law?*

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Abstract

This paper argues that the distribution of the coefficients of the regular continued fraction should be considered for each algebraic number of degree > 2 separately. For random numbers the coefficients are distributed by the Gauss-Kuzmin distribution (also called Khinchin's law). We apply the Kullback Leibler Divergence (KLD) to show that the Gauss-Kuzmin distribution does not fit well for algebraic numbers of degree > 2. Our suggestion to truncate the Gauss-Kuzmin distribution for finite parts fits slightly better, but its KLD is still much larger than the KLD of a random number. We consider differences regarding Khinchin's constant and Khinchin's approximation speed between random and algebraic numbers and conclude that laws concerning the random numbers do not automatically carry over to the algebraic numbers.

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1 Introduction

The coefficients of the regular continued fraction of a uniformly distributed random number follow Khinchin's Law. Already Gauss had guessed this and Kuzmin (1928), cf. (Khinchin, 1963, p. 64, theorem 33) proved it.

It is an open question whether algebraic numbers of degree > 2 also follow Khinchin's Law. One major aim of Serge Lang's classical work is to show that this is the case as the quote from the backcover illustrates (Lang, 1995, p. 138):

> One general idea is that algebraic numbers will exhibit a behaviour that is the same as almost all numbers in a probabilistic sense, except under very specific structural conditions, namely quadratic numbers. Results for almost all numbers (due to Khinchin) show an interplay between calculus and number theory.

Lang & Trotter (1972) made χ^2-tests and concluded that the results are as expected. (Bombieri & van der Poorten, 1975, p. 141) even go as far as to say:

> There is no reason to believe that the continued fraction expansions of nonquadratic algebraic irrationals generally do anything other than to faithfully follow Khinchin's Law as detailed below. Indeed experiment suggests that this is even true for parts, short relative to the length of the period, of the expansions of quadratic irrationals.

We argue that matters are different for the following reasons:

1. Parts of the period of quadratic irrationals show that the χ^2 tests are not reliable.

2. The Kuzmin distribution is unbounded, while the coefficients of finite parts of the regular continued fraction of algebraic numbers are bounded due to Liouville's theorem. Furthermore, it is known that the Liouville bounds can be improved. It can presently not be ruled out that all coefficients of the regular continued fraction of algebraic numbers are bounded. This is an old, open question.

3. Since this is an open question, an account is needed that can also deal with the case of bounded coefficients. We suggest such an account below.

4. Why should *all* algebraic numbers with degree > 2 follow Khinchin's Law? It is more likely that each number follows its own distribution. Our suggestion will consider this.

5. The convergence behaviour of the geometric means of the coefficients of the regular continued fractions is very different for each algebraic number. For a uniformly distributed random number, however, it converges against Khinchin's constant. So, it is unlikely to infer the distribution of the coefficients for algebraic numbers from the distribution of random numbers.

6. To mix calculus with number theory is problematic. Real numbers are equivalence classes of Cauchy sequences, which are approximated according to calculus. Number theory, in contrast, considers different types of numbers due to their specific definitions. Algebraic numbers, for example, are rooted in algebraic equations, which connects them to rational numbers. There is nothing random about an algebraic number; rather its specific properties can be studied by the equation that defines it.

This paper is structured as follows. In section 2, we discuss continued fractions of algebraic numbers. Section 3 measures the goodness of fit for our newly proposed truncated Gauss-Kuzmin distribution in comparison to the standard Gauss-Kuzmin distribution. Section 4 shows that each algebraic number seems to have its own special behaviour regarding Khinchin's constant. Section 5 lists conjectures regarding the role of Khinchin's Law, Khinchin's constant and Khinchin's approximation speed for algebraic numbers. Section 6 draws the conclusion.

2 Continued Fractions of Algebraic Numbers

This section considers the distribution of the coefficients of regular continued fractions of algebraic numbers with degree > 2. Lang & Trotter (1972) aim to show that these coefficients follow a Gauss-Kuzmin distribution. The distribution is defined as follows. Consider a continued fraction expansion of a random number x uniformly distributed in $(0, 1)$:

$$x = \frac{1}{k_1 + \frac{1}{k_2 + \ldots}}.$$

Then the following holds asymptotically for the distribution of the coefficients k_n:

$$\lim_{n \to \infty} P(k_n = k) = -\log_2(1 - \frac{1}{(k+1)^2}).$$

This distribution is known to be the Gauss-Kuzmin distribution. The approximation of the Gauss-Kuzmin distribution is an asymptotic result for uniformly distributed random numbers. Lang & Trotter (1972) claim that this result carries over to algebraic numbers of degree > 2.

However, the Gauss-Kuzmin distribution is unbounded by definition while the coefficients of finite parts of the continued fraction expansion are bounded by Liouville's theorem.

Liouville's theorem says that for an algebraic number a of a degree $n > 2$ there exists a positive constant $C(a)$ with

$$|a - \frac{p}{q}| > \frac{C(a)}{q^n}$$

for all natural numbers p and q.

Liouville's bound is very rough. In general, each algebraic number of degree > 2 has its own bound. Because the Thue Siegel Roth theorem (proven by Roth (1955)) provides only an upper bound for a measure of the speed of the best rational approximation, it is an open question to what extent it can be improved for specific algebraic numbers with effectively computable bounds. Voutier (2007) investigates this for $\sqrt[3]{2}$ and mentions the following famous result by Korobov (1990)

$$|\sqrt[3]{2} - \frac{p}{q}| > \frac{1}{q^{2.5}}$$

for all natural numbers p and q with the exception of 1 and 4 for q.

Hence, effectively computable bounds exist, which are much sharper than Liouville's. These sharper bounds could be used to truncate the Gauss-Kuzmin distribution for algebraic numbers, logarithms of rational numbers and other numbers for which the method with the hypergeometric series, explained by Voutier (2007), applies. It can even be

conjectured that all periods have such bounds. This is an open question (Waldschmidt, 2006, p. 437, question 2). A period is a number that can be expressed as an integral of an algebraic function over an algebraic domain. It is difficult to obtain general results because each period has its very own behaviour. It seems to be especially difficult to obtain an optimal lowest upper bound. For certain periods like $\sqrt[3]{2}$ Voutier (2007), $\zeta(2)$ Zudilin (2014) and π Zeilberger & Zudilin (2020) this has been dealt with many times, always slightly improving the results and in all cases research is still going on. The history for π is especially fascinating. Mahler (1953) started it with the upper bound 42 for the effective irrationality measure (in Korobov's formula this is 2.5 as exponent of q) and currently Zeilberger & Zudilin (2020) hold the record with 7.103205334137. This picture is completely different from the random numbers. It seems to be more interesting to study the differences from random numbers than to conjecture their similar behaviour. Hence, it is no wonder that this is an active research area.
(Waldschmidt, 2006, p. 437) poses a further problem:

> A more ambitious goal would be to ask whether real or complex periods behave, from the point of view of Diophantine approximation by algebraic numbers, like almost all real or complex numbers.

We conjecture that periods do not follow Khinchin's Law. In addition to study differences between random and algebraic numbers, it is an important, though even more difficult, endeavour to investigate differences between random numbers and periods. At least for certain classes of periods, it should be possible to obtain results.
Since the Gauss-Kuzmin distribution does not hold for finite parts of the continued fractions of algebraic numbers, we suggest to take this into account by a truncated version of the Gauss-Kuzmin distribution. The truncated Gauss-Kuzmin distribution for a finite part is defined by the probability function

$$P_{truncK}(k_n = k) = \frac{P_K(k)}{1 - \log_2(\frac{2+maxn}{1+maxn})} \tag{1}$$

$P_K(k)$ denotes the probability function of the standard Gauss-Kuzmin distribution for $k \leq maxn$ and 0 for $k > maxn$, where $maxn$ is the maximum of the coefficients of the regular continued fraction of the algebraic number for that finite part.

By truncating the Gauss-Kuzmin distribution, we therefore open up the possibility that the coefficients of the regular continued fraction representation of algebraic numbers are bounded in general. For infinite unbounded continued fractions our distribution approaches the standard Gauss-Kuzmin distribution, which is thus also embedded in our account.

3 Measuring the Goodness of Fit of the Truncated Kuzmin Distribution

Lang & Trotter (1972) used χ^2 goodness of fit tests to show that the coefficients of continued fractions of algebraic numbers follow a Gauss-Kuzmin distribution. They draw their conclusions from non-rejections of the χ^2 tests. However, as is well known, the non-rejection of a null hypothesis is no proof for the hypothesis to be true since a type II error with unknown error probability occurs. In addition to this, the χ^2 test is known to be not very reliable. It is justified mainly because it is easy to use, particularly, when the access to computer power is limited.

We, therefore, revisit this question by using the Kullback Leibler Divergence (KLD) instead. We do not merely test for one possible distribution whether it fits the data or not but also compare two different distributions in how they fit the data.

In the following, log is used and 2 is the base throughout. The Kullback Leibler Divergence for the discrete distributions P and Q is defined as

$$KLD(P,Q) = \sum_x P(x)\log(\frac{P(x)}{Q(x)}). \tag{2}$$

We have calculated the KLD for 1000 coefficients for algebraic numbers and the standard Gauss-Kuzmin distribution. Table 1 shows the results for some roots of 2.

The values in Table 1 can be compared to the KLDs of the regular continued fractions of pseudo random numbers. We choose six pseudo random numbers with 1000 decimal digits and calculated the KLDs rounded to 4 digits: 0.0836, 0.0603, 0.0836, 0.0573, 0.0802, 0.0718. This, as well as our calculations of the KLDs for further algebraic numbers, indicates that the Gauss-Kuzmin distribution fits much better to the pseudo random numbers than to the algebraic numbers, which gives reason to believe that the

Algebraic number	KLD
$\sqrt[3]{2}$	0.0955
$\sqrt[4]{2}$	0.0744
$\sqrt[5]{2}$	0.0905
$\sqrt[6]{2}$	0.1103
$\sqrt[7]{2}$	0.1117
$\sqrt[8]{2}$	0.0931

Table 1: KLD rounded to 4 digits

distribution of the algebraic numbers is not the same.

Therefore, we suggest to truncate the Gauss-Kuzmin distribution as defined in equation (1) for finite parts of the regular continued fraction of each algebraic number with a different bound using the properties of the specific number. Note that this truncation of the distribution is different from the truncation of the Gauss-Kuzmin law, which is used by Hensley (1988).

Table 2 provides our calculations of the KLDs for the truncated distribution for 1000 coefficients of the numbers from Table 1.

Algebraic number	KLD
$\sqrt[3]{2}$	0.0953
$\sqrt[4]{2}$	0.0730
$\sqrt[5]{2}$	0.0884
$\sqrt[6]{2}$	0.1102
$\sqrt[7]{2}$	0.1114
$\sqrt[8]{2}$	0.0920

Table 2: KLD for the truncated Kuzmin distribution rounded to 4 digits

For the same 6 pseudo random numbers as before we obtain the KLDs 0.0832, 0.0592, 0.0825, 0.0563, 0.0787, 0.0669. The KLDs of the truncated Gauss-Kuzmin distribution are always lower than those of the Gauss-Kuzmin distribution. This indicates that the Gauss-Kuzmin distribution is not the distribution of the coefficients of the regular continued fraction of an algebraic number. Note that even the truncated Gauss-Kuzmin distribution still has a quite large KLD compared to the random numbers. So, it seems likely to consider each number's own distribution.

The KLDs of the truncated Gauss-Kuzmin distribution also indicates that the Gauss-Kuzmin distribution itself is slightly too large for coefficients of algebraic numbers with degree > 2 because the KLDs of the truncated Gauss-Kuzmin distribution are always better. This can be also concluded from the fact that both, the standard and the truncated Gauss-Kuzmin distribution, are only defined for positive numbers. Thus, the probability mass of the standard Gauss-Kuzmin distribution of values greater than our truncation is redistributed over the finite interval of our truncated version according to the Gauss-Kuzmin distribution.

Furthermore, it is not even clear whether the truncated distributions converge to the Gauss-Kuzmin distribution, because it is still an open question whether the coefficients are bounded or not. Adamczewski & Bugeaud & Davison (2006) discuss this and give more sources.

For these reasons, we suggest to use the truncated Gauss-Kuzmin distribution. This might also be interesting for π, $\log(2)$ and other numbers, which appear naturally in analysis in the place of the algebraic numbers of degree > 2.

This is related to Lang's open conjecture that the approximation speed $B(n)^2 \ln(B(n))$ is the best possible for his classical numbers as defined in (Lang, 1971, p. 635). This means that if $A(n)/B(n)$ with $B(n) > 0$ is a rational sequence converging to the classical number a, then for every $\varepsilon > 0$ $|a - A(n)/B(n)|B(n)^2(\ln(B(n))^{1+\varepsilon}$ is always bounded away from 0 (Lang, 1971, p. 664).

One may go even further and conjecture that this speed is already too fast and that $|a - A(n)/B(n)|B(n)^2(\ln(B(n))$ is always bounded away from 0 for Lang's classical numbers. For quadratic irrationals, for which $B(n)^2$ is the well-known approximation speed, and for Euler's number (Lang, 1995, p. 76, theorem 5) this is proven but otherwise it is an open question.

4 Khinchin's Constant

Another indication that each algebraic number of degree > 2 should be investigated separately is Khinchin's constant $K0$. (Bailey & Borwein & Crandall, 1997, p. 423) write:

> It is remarkable that, even though a random fraction's limiting geometric mean exists and furthermore equals the Khinchin constant with probability

one, not a single explicit real number (e.g., a real number cast in terms of fundamental constants) has been demonstrated to have elements whose geometric mean equals K0.

Let us abbreviate the truncated Gauss-Kuzmin distributions at $maxn$ by $GKT(maxn)$. To test the hypothesis that finite parts of the regular continued fraction of algebraic numbers of degree > 2 are distributed with $GKT(maxn)$ we first calculate the value $KC(maxn)$ of the limit of the geometric means of the coefficients of the regular continued fraction of random numbers distributed with $GKT(maxn)$:

$$KC(1) = 1,\ KC(maxn) = 2^{GKT(2)} \cdot \ldots \cdot maxn^{GKT(maxn)} \text{ for } maxn \geq 2.$$

This sequence converges for $maxn \to \infty$ monotonously increasing to Khinchin's constant $K0$ as expected.

This theoretical result for random numbers can now be compared to numerical calculations of the geometrical mean for algebraic numbers. As Table 3 shows for our test cases, the expectation is not confirmed since the geometric mean is even larger than $K0$ in several cases.

Number	Geometric mean
random number	2.685
$\sqrt[3]{3}$	2.735
$\sqrt[4]{3}$	2.742
$\sqrt[5]{3}$	2.671
$\sqrt[6]{3}$	2.696
$\sqrt[7]{3}$	2.711
$\sqrt[8]{3}$	2.692

Table 3: Geometric mean for the first 10000 coefficients

To sum up, algebraic numbers behave differently from random numbers and should be investigated separately.

5 The Role of Randomness

The following theorems can be proven for random numbers by probabilistic means:

T1: Khinchin's Law holds for the distribution of the coefficients of the regular continued fraction (Khinchin, 1963, p. 92f.).

T2: Khinchin's constant is the limit of the geometric means of those coefficients (Khinchin, 1963, p. 93).

T3: Khinchin's approximation speed $B(n)^2 \ln(B(n))$ is reached (Khinchin, 1963, p. 69). The convergents of the random number are $A(n)/B(n)$.

But what role do these theorems play for algebraic numbers of degree > 2?
We conjecture:

C1: Khinchin's Law is near the real distribution, but it is not the distribution itself.

C2: Khinchin's constant is near the geometric means for large n.

C3: Khinchin's approximation speed $B(n)^2 \ln(B(n))$ is an unreachable upper bound.

According C1, there is an upper bound for the coefficients for finite parts. And even when the Gauss-Kuzmin distribution is truncated for finite parts, it is still only near the real distribution. Using the KLDs, it might be interesting to investigate further what "near" means exactly in this context.
According C2, it seems to be very difficult to be more precise here.
C3 means that if $A(n)/B(n)$ with $B(n) > 0$ is a rational sequence converging to the algebraic number a, then $|a - A(n)/B(n)|B(n)^2(\ln(B(n))$ always diverges to infinity. C3 can even be generalised for periods and Lang's classical numbers. It can be proven for quadratic irrationals and Euler's number. For algebraic numbers of degree > 2, the question is open. It is even open, whether the coefficients are bounded at all. Our calculation seems to indicate that the Gauss-Kuzmin distribution is slightly too large, which would indicate that Khinchin's speed is too fast for algebraic numbers, which agrees with the experimental evidence reported by (Lang & Trotter, 1972, p. 117):

> The tables [for the first 3000 terms of the continued fractions of the cubic numbers] suggest that the type may in fact not be bigger than a constant times the logarithm, and may even be of an order of magnitude smaller than the logarithm.

Another indication is the connection of the speed for algebraic numbers with the solutions of diophantine equations. Experimental evidence shows that speed is slow, when only small solutions exist (Smart, 1998, p. 135).

C3 can be weakened to the effect that Khinchin's speed is an upper bound for periods and Lang's classical numbers. It can be even further weakened to apply Lang's Conjecture (Soundararajan, 2011, p.52, Conjecture 7) to periods and Lang's classical numbers. In that case it is in line with Khinchin's convergence principle (Khinchin, 1963, Theorem 32, p.69). Yet, there are no counterexamples to C3 known. So, we can conjecture it as a challenge for future research.

The conjectures compare, in a way, real randomness with pseudo randomness and state that the pseudo randomness applying to specific numbers defined by equations is near real randomness but does not reach it exactly. This is known from other generators of pseudo random numbers as well.

6 Conclusion

This paper revisits the question which properties of random numbers carry over to algebraic numbers. For random numbers the coefficients of the regular continued fraction follow a Gauss-Kuzmin distribution. For algebraic numbers of degree > 2 this seems implausible because the coefficients of finite parts are bounded by Liouville's theorem whereas the Gauss-Kuzmin distribution is unbounded. We therefore propose a new truncated Gauss-Kuzmin distribution to model the distribution of the coefficients of finite parts of the regular continued fraction of algebraic numbers of degree > 2. We apply the Kullback Leibler Divergence to show that our truncated Gauss-Kuzmin distribution gives a better fit than the standard Gauss-Kuzmin distribution. This finding is underpinned by simulation results for a variety of algebraic numbers of degree > 2. Yet, the KLDs are still quite large. Furthermore, the convergence behaviour of the geometric means is very different. It is not even clear, whether the means converge to Khinchin's constant or not. Likewise, it is still an open question, whether Khinchin's speed applies to algebraic number of degree > 2. These questions are also interesting to consider for larger classes of numbers like periods or Lang's classical numbers. Probably each such number has its own distribution. In any case, great care is required when laws from the world of randomness are carried over to the world of laws.

References

Adamczewski, B., Bugeaud, Y. and Davison, L. (2006): "Continued Fractions and Transcendental Numbers", *Annales de l'institut Fourier* 56, 2093–2113.

Bailey, D., Borwein, J. and Crandall, R. (1997): "On the Khinchin Constant", *Mathematics of Computation* 66(217), 417–431.

Bombieri, E. and van der Poorten, A. (1975): "Continued Fractions of Algebraic Numbers", in: Baker (ed.), *Transcendental Number Theory*, Cambridge University Press, Cambridge, 137-155.

Hensley, D. (1988): "A Truncated Gauss-Kuzmin Law", *Transactions of the American Mathematical Society* 306(1), 307-327.

Khinchin, A. (1963): *Continued Fractions*, P.Noordhoff, Groningen.

Korobov, A. (1990): *Continued Fractions and Diophantine Approximations*, Candidate's Dissertation, Moscow State University.

Kuzmin, R. O.(1928): "On a Problem of Gauss", *Doklady Akademii Nauk SSSR*, 375–380.

Lang, S. (1971): "Transcendental Numbers and Diophantine Approximation", *Bulletin of the American Mathematical Society* 77(5), 635-677.

Lang, S. and Trotter, H. (1972): "Continued Fractions for some Algebraic Numbers", *Journal für die reine und angewandte Mathematik* 255, 112–134; Addendum: ibid. 219–220.

Lang, S. (1995): *Introduction to Diophantine Appoximations*, new expanded edition, New York/Berlin/Heidelberg, Springer.

Mahler, K. (1953): "On the Approximation of π", *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings* Series A 56(1), 30–42.

Roth, K. (1955): "Rational Approximations to Algebraic Numbers and Corrigendum", *Mathematika* 2, 1-20 and 168.

Smart, N. (1998): *The Algorithmic Resolution of Diophantine Equations*, London Mathematical Society Student Texts 41, London.

Soundararajan, K. (2011): *Transcendental Number Theory*, Math 249A Fall 2010, http://math.stanford.edu/ ksound/TransNotes.pdf.

Voutier, P. (2007): “Rational Approximation to $\sqrt[3]{2}$ and other Algebraic Numbers Revisited”, *Journal des Nombres de Bordeaux* 19, 263-288.

Waldschmidt, M. (2006): “Transcendence of Periods: The State of the Art”, *Pure and Applied Mathematics Quarterly* 2(2), 435—463.

Zeilberger, D. and Zudilin, W. (2020): “The Irrationality Measure of π is at most 7.103205334137”, *Moscow Journal of Combinatorics and Number Theory* 9(4), 407–419.

Zudilin, W. (2014): “Two Hypergeometric Tales and a new Irrationality Measure of $\zeta(2)$”, *Annales mathématiques du Québec* 38(1), 101–117.