Roth's Theorem implies a Weakened Version of the ABC Conjecture for Special Cases[*]

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Abstract

Enrico Bombieri proved that the ABC Conjecture implies Roth's theorem in 1994. This paper concerns the other direction. In making use of Bombieri's and Van der Poorten's explicit formula for the coefficients of the regular continued fractions of algebraic numbers, we prove that Roth's theorem implies a weakened non-effective version of the ABC Conjecture in certain cases relating to roots.

1 Introduction

The ABC Conjecture is one of the most famous open problems in number theory. Masser and Oesterle conjectured it in Masser (1985) and Oesterle (1988). We investigate connections to Roth's theorem, which was proven by Roth (1955). Bombieri (1994) proved that the ABC Conjecture implies Roth's theorem. We consider the other direction under weakened assumptions for special cases.

2 The ABC Conjecture

To formulate the ABC Conjecture we need the following definition:

Definition 2.1. *For a positive integer a, $rad(a)$ is the product of the distinct prime factors of a.*

Definition 2.2 (ABC Conjecture)**.** *For every positive real number ε, there exists a constant K_ε such that for all triples (a, b, c) of coprime positive integers, with $a + b = c$:*

$$c < K_\varepsilon \cdot rad(abc)^{1+\varepsilon}.$$

Note that this version of the ABC Conjecture is not effective. This is so because only the existence of a constant K_ε is claimed without specifying how to calculate K_ε for given ε.

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Explicit effective versions which specify K_ε, however, are used in proofs related to equations of form $a^m + b^n = c^k$. In such proofs, the use of an effective ABC Conjecture can ignore prime factorisation and, thus, $rad(abc)$. Instead, it suffices to refer to abc without exponents. This motivates our definition of a "weakened effective version of ABC" (WEABC).

Definition 2.3 (WEABC). *For any fixed $\varepsilon > 0$ there is a constant K_ε such that for all triples (a,b,c) of coprime positive integers and $m,n,k \in \mathbb{N}$, with $a^m + b^n = c^k$:*

$$c^k < K_\varepsilon \cdot (abc)^{1+\varepsilon}.$$

This definition differs from the literature where effective versions of the ABC Conjecture for a fixed $\varepsilon > 0$ are called "weak ABC Conjecture", e.g., by (Bombieri & Gubler, 2006, p. 403, example 12.2.3).

Fermat's last theorem follows immediately from WEABC for $n \geq 6$ (Bombieri & Gubler, 2006, p.403, example 12.2.4):

Theorem 2.4. *The WEABC with $\varepsilon = 1$ and $K_\varepsilon = 1$ implies Fermat's Theorem for $n \geq 6$.*

Proof 1. *Let x, y and z be coprime positive integers satisfying $x^n + y^n = z^n$. With $a = x^n$, $b = y^n$, $c = z^n$ we get $z^n \leq rad(xyz)^2 \leq (xyz)^2 \leq z^6$. As $z > 1$ the desired inequality $n \leq 5$ follows.* $\square$

Likewise, Catalan's conjecture for large n and m can be proven from WEABC.

Furthermore, theorems that have been proven independently of the ABC Conjecture and that could be proven by the ABC Conjecture like Wiles (1995) proof of Fermat's conjecture, WEABC suffices to prove those theorems. So it is enough to get rid off the exponent n.

If we do not refer to a specified K_ε, we speak of a "weakened non-effective ABC Conjecture" (WNEABC).

Definition 2.5 (WNEABC). *For every $\varepsilon > 0$, there exists a constant K_ε for all triples (a,b,c) of coprime positive integers and $m,n,k \in \mathbb{N}$, with $a^m + b^n = c^k$:*

$$c^k < K_\varepsilon \cdot (abc)^{1+\varepsilon}.$$

Such a conjecture can be proven from Roth's theorem because it is not necessary to find estimates for the prime numbers, which is extremely difficult. A generalised version of Ridout's theorem would even prove the full ABC Conjecture for these cases, but this seems to be very hard.

Theorem 2.6. *Roth's Theorem: Let a be an algebraic number. Then for every $\varepsilon > 0$, there exists a constant K_ε such that for all positive integers p and q:*

$$\left| a - \frac{p}{q} \right| > \frac{K_\varepsilon}{q^{2+\varepsilon}}.$$

abc Equation	Root	Founder
$2 + 23^5 = 9^5 \cdot 109$	$\sqrt[5]{109}$	Reyssat
$5^{56} \cdot 245983 = 2^{15} 3^{77} \cdot 11 \cdot 173 + 2543^4 \cdot 182587 \cdot 2802983$	$\sqrt[56]{245983}$	Bonse
$5^4 \cdot 7 = 23^7 + 1$	$\sqrt[4]{7}$	De Weger

Table 1: Particular cases of abc Equations and Roots

Proof 2. *See Roth (1955).* □

Roth's theorem is not effective. Therefore, only non-effective ABC cases can follow from it. Roth's theorem can probably be sharpened. It is even an open problem whether the coefficients of the regular continued fraction of algebraic numbers of degree > 2 are bounded or not. If Roth's theorem can be sharpened then the WNEABC can be sharpened as well.

Counterexamples for the ABC Conjecture with $\varepsilon = 0$ are known. But these counterexamples do not result from algebraic equations. Therefore, the coefficients of the regular continued fractions of algebraic numbers of degree greater than 2 can nonetheless be bounded.

Definition 2.7. *A triple of coprime integers $a + b = c$ is an ABC hit, if $rad(abc) < c$ holds and the quality of a triple is given by the first $quality(a, b, c) = \frac{ln(c)}{ln(rad(abc))}$. As $c > 1$ it is clear that abc is larger than 1.*

$2 + 23^5 = 9^5 \cdot 109$ is the ABC hit with the highest quality so far found. Its quality is approximately 1.62991. This hit results from the 3rd convergent to $\sqrt[5]{109}$ with regular continued fraction $[2; 1, 1, 4, 77733, \ldots]$. The fraction $\frac{23}{9}$ is a good approximation because 77733 is very large and very near the absolute Liouville bound.

Thus, ABC hits are connected with diophantine approximations. We want to go further in this direction. Each number of the form $\sqrt[s]{k}$ leads to a set of *abc* equations due to the convergents of the root. However, the resulting a, b and c are not necessarily coprime. This is not a restriction because the ABC Conjecture implies Roth's theorem, as proven by Bombieri (1994). We will show by Theorem 2.10 that WNEABC is also valid for these equations even when the resulting a, b and c are not coprime.

Each root $\sqrt[s]{k}$ is directly connected with a set of *abc* equations.

Definition 2.8. *Resulting equations for $\sqrt[s]{k}$:= Let $\frac{p_n}{q_n}$ be a convergent of $\sqrt[s]{k}$ and $d_n = kq_n^s - p_n^s$. If $d_n > 0$ then the resulting equation is $d_n + p_n^s = kq_n^s$, otherwise it is $-d_n + kq_n^s = p_n^s$.*

p_n and q_n are always coprime, but k and d_n may have common prime factors with the other factors of the equations.

From the ABC record hits known until now most hits have resulting equations, cf. Table 1 for the most famous ones. It is remarkable that several of the best ABC hits have very small a, b, c.

For our WEABC, a 2nd quality results in these cases, which only considers the approximation speed and not the prime factorisation of the convergents:

Definition 2.9. *The 2nd quality(a,b,c) equals $\frac{ln(q_n{}^s k)}{ln(d_n q_n k p_n)}$ if $d_n > 0$, and $\frac{ln(p_n{}^s)}{ln(-d_n q_n k p_n)}$ if $d_n < 0$.*

The 2nd quality is always smaller or equal to the 1st quality. There are already ABC hits regarding the 2nd quality alone, in the sense that the 2nd quality is larger than 1. The WNEABC is known to be equivalent to the conjecture that the 2nd quality has Limes Superior ≤ 1 for the resulting equations of the convergents of a root.

The main aim of this paper is to show how to prove the following theorem:

Theorem 2.10. *The WNEABC is valid for all resulting equations for $\sqrt[s]{k}$.*

3 Proof of Theorem 2.10

d_n, p_n and q_n need not to be coprime. But often they will be coprime. So not all cases are "real ABC cases", but the inequality of the WNEABC is always valid.

We first illustrate how to prove Theorem 2.10 by proving it for quadratic irrationals $\sqrt[2]{k}$. In these cases all d_n are 1 or -1. Furthermore, the coefficients b_n are periodic and therefore bounded. But this latter fact is not even needed.

Proof 3. *Case 1: $d_n = 1$. According Definition 2.8, $1 + p_n{}^2 = k \cdot q_n{}^2$. So for the 2nd quality the following results:*

$$2nd\ quality = \frac{ln(q_n{}^2 \cdot k)}{ln(p_n \cdot k \cdot q_n)} = \frac{2 \cdot ln(q_n) + ln(k)}{ln(p_n) + ln(k) + ln(q_n)} = \frac{2 + \frac{ln(k)}{ln(q_n)}}{1 + \frac{ln(k)}{ln(q_n)} + \frac{ln(p_n)}{ln(q_n}} = \frac{2 + \frac{ln(k)}{ln(q_n)}}{1 + \frac{ln(k)}{ln(q_n)} + \frac{ln(\frac{p_n}{q_n}) + ln(q_n)}{ln(q_n)}}$$

and this converges to 1 as all terms are bounded apart from the unbounded $ln(q_n)$. Case 2: $d_n = -1$ results similarly in

$$\frac{2}{2 + \frac{ln(k)}{ln(p_n)} + \frac{ln(\frac{q_n}{p_n})}{ln(p_n)}}$$

which converges to 1 as well. □

When there are laws for the prime factors of the denominator of the quality they can be taken into account for specifying a 3rd quality. For example, for $\sqrt[2]{2}$ with $d_n < 0$ this can be done with the prime factor 2 since q_n is always even.

This can be seen from the recurrence relations of the coefficients of the regular continued fraction. $q_1 = 2$ is even and $q_{n+2} = 2q_{n+1} + q_n = 5q_n + 2q_{n-1}$, therefore all q_n are even, if n is odd. This leads to the following specification of the 3rd quality of this case:

$$3rd\ quality = \frac{ln(p_n{}^2)}{ln(p_n q_n)} = \frac{2}{2 + \frac{ln(\frac{q_n}{p_n})}{ln(p_n)}} = \frac{2}{2 + \frac{ln(q_n)}{ln(p_n)}) - 1}.$$

The 3rd quality converges monotonously decreasing to 1 because the denominator is monotonously increasing and so the maximum of the 3rd quality is approximately 1.226 at the resulting equation $1 + 8 = 9$, which is a real ABC equation. In this case all resulting equations are real ABC equations as q_n and p_n are coprime and q_n is even so that p_n must be odd and the prime factor $k = 2$ cannot interfere.

The 3rd quality is always larger than the 2nd quality.

The laws for the prime factorisation of the convergents, however, are not sufficient to get the full ABC inequality with $rad(abc)$. For example, it is known that from the resulting Pell numbers in the denominators of the convergents of $\sqrt[2]{2}$ the only squares, cubes, or any higher power of an integer are 0, 1, and $169 = 13^2$ (for more details see Cohn (1996)). An analogous result for Fibonacci numbers has been obtained by Bugeaud (2006).

The main tool for the following proofs is the explicit formula for the coefficients of the regular continued fractions of algebraic numbers by (Bombieri & van der Poorten, 1975, p.151, Theorem 3, formula (13)), in which $f(x)$ is the minimal polynomial of the algebraic number. The formula connects d_n and b_n so that WNEABC can be proven from Roth's theorem, which has no direct connection to d_n. In the main term of the formula, $\frac{f'(x)}{f(x)}$ leads to d_n in the denominator. This holds because $\frac{f'(x)}{f(x)}$ results in $\frac{f'(\frac{p_n}{q_n})}{f(\frac{p_n}{q_n})} = \frac{q_n^s s p_n^{s-1}}{d_n q_n^{s-1}}$ for the convergents with $f(x) = x^s - k$ being the minimal polynomial of the sth root of k.

For the 3rd degree the error term of the formula (Bombieri & van der Poorten, 1975, p.151, Theorem 3, formula (13)) can be ignored due to the effective bound by Liouville's theorem. For higher order it can always be controlled, when n is large enough, which suffices here to prove WNEABC.

To illustrate the use of Bombieri's and van der Poorten's formula, we first provide a criterion for an ABC hit regarding the 2nd quality of $\sqrt[3]{2}$.

Theorem 3.1. *If $b_{n+1} > 6$ then the resulting equation from $\frac{p_n}{q_n}$ is an ABC hit even regarding the 2nd quality.*

Proof 4. *Let $d_n > 0$. Then the ABC hit condition for the 2nd quality is*

$$\frac{\ln(p_n^3)}{\ln(2 \cdot p_n \cdot q_n \cdot d_n)} > 1.$$

Since the logarithm is monotonously increasing, the last equation is equivalent to

$$p_n^3 > 2 \cdot p_n q_n d_n.$$

The previous equation, in turn, is equivalent to

$$\frac{p_n^2}{2q_n} > d_n.$$

From (Bombieri & van der Poorten, 1975, p.151, Theorem 3, formula (13)), we have that

$$b_{n+1} < \frac{3p_n^2}{q_n d_n}.$$

Therefore, we find that

$$d_n < \frac{3p_n^2}{q_n b_{n+1}}.$$

Hence, the ABC 2nd quality hit condition $p_n^3 > 2p_n q_n d_n$ is satisfied for $b_{n+1} > 6$ because the inequality $\frac{3 \cdot p_n^2}{q_n b_{n+1}} < \frac{p_n^2}{2 \cdot q_n}$ holds in this case. $\square$

We now illustrate the proof of Theorem 2.10 for $\sqrt[3]{2}$. We start with the case that the coefficients b_n are bounded. In this case, $\varepsilon = 0$ can even be chosen in WNEABC.

Theorem 3.2. *The WNEABC is valid for all resulting equations for $\sqrt[3]{2}$.*

Proof 5. *The explicit Bombieri / van der Poorten formula for $\sqrt[3]{2}$ for the coefficients of the regular continued fraction is*

$$b_{n+1} = \lfloor (\frac{(-1)^{n+1}}{q_n} \frac{3p_n^2}{-d_n} - \frac{q_{n-1}}{q_n}) \rfloor.$$

Let the natural number K be the bound of b_n. By the formula of (Bombieri & van der Poorten, 1975, p.151, Theorem 3, formula (13)) the following follows from $b_{n+1} < K$:

$$\frac{3p_n^2}{q_n|d_n|} < (K+2) \quad \Leftrightarrow \quad \frac{3p_n^2}{q_n(K+2)} < |d_n|.$$

To get WNEABC with $\varepsilon = 0$ a new constant L must be found such that

$$2q_n^3 < 2 \cdot L \cdot p_n q_n |d_n| \ \ if \ d_n < 0;$$
$$p_n^3 < 2 \cdot L \cdot p_n q_n d_n \ \ if \ d_n > 0.$$

This leads to

$$\frac{q_n^2}{p_n L} < |d_n|$$

and

$$\frac{p_n^2}{2q_n L} < d_n.$$

Thus L must be chosen to satisfy both

$$\frac{q_n^2}{p_n L} < \frac{3p_n^2}{q_n(K+2)}$$

and

$$\frac{p_n^2}{2q_n L} < \frac{3p_n^2}{q_n(K+2)}.$$

This is possible because $\frac{p_n}{q_n}$ converges to $\sqrt[3]{2}$.
It follows that WNEABC follows when $\varepsilon = 0$.
To prove that WNABC hold when $\varepsilon > 0$, we need the following Lemma:

Lemma 3.3. *Roth's Theorem implies that $\frac{p_n^\varepsilon}{b_{n+1}}$ is bounded away from zero for every $\varepsilon > 0$.*

Proof 6. *This Lemma is proven by the following equations:*

$$0 < K < |a - \frac{p_n}{q_n}|q_n^{2+\varepsilon} < |\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}|q_n^{2+\varepsilon} = \frac{1}{q_n q_{n+1}}q_n^{2+\varepsilon}$$

$$= \frac{q_n}{q_{n+1}}q_n^\varepsilon = \frac{q_n}{b_{n+1}q_n+q_{n-1}}q_n^\varepsilon = \frac{1}{b_{n+1}+\frac{q_{n-1}}{q_n}}q_n^\varepsilon \leq \frac{q_n^\varepsilon}{b_{n+1}}$$

$\square$

The weakened non-effective ABC condition is

$$K_\varepsilon(2 \cdot d_n p_n q_n)^{1+\varepsilon} \quad \geq \quad K_\varepsilon\left(\frac{6 p_n^3}{b_{n+1}+2}\right)^{1+\varepsilon} \quad \geq \quad K_\varepsilon \cdot 6^{1+\varepsilon} p_n^3 \left(\frac{p_n^\varepsilon}{b_{n+1}+2}\right)\left(\frac{p_n^2}{b_{n+1}+2}\right)^\varepsilon$$

The other case $d_n > 0$ can be proven analogously.

Because $\left(\frac{p_n^\varepsilon}{b_{n+1}+2}\right)$ is bounded away from zero by Roth's theorem and $\left(\frac{p_n^2}{b_{n+1}+2}\right)$ is bounded away from zero by the formula in (Bombieri & van der Poorten, 1975, p.151, Theorem 3, formula (13)), b_{n+1} behaves like $\frac{p_n}{d_n}$. Therefore, $K_\varepsilon > 0$ can be chosen, so that the weakened non-effective ABC condition can be satisfied, which concludes the proof. $\square$

To apply this proof to other roots of the form $\sqrt[s]{k}$, all that has to be done is to exchange numbers. Therefore, Proof 5 for $\sqrt[3]{2}$ generalises to all roots $\sqrt[s]{k}$, which proves Theorem 2.10. However, it has to be noticed that each root has its own K_ε, which is ensured by Roth's theorem. Note that for higher degree in (Bombieri & van der Poorten, 1975, p.151, Theorem 3, formula (13)) formula (13) is valid and the error term can be ignored for *n* large enough. So, our case can always be solved.

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