Testing for fractional cointegration in subsamples by allowing for structural breaks

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Abstract

In this paper, tests for fractional cointegration that allow for structural breaks in the long-run equilibrium are proposed. Traditional cointegration tests cannot handle shifts in fractional cointegration relationships, a limitation addressed here by allowing for a time-dependent memory parameter for the cointegration error. The tests are implemented by taking the extremum of a residual-based fractional cointegration test applied to different subsamples of the data. The subsampling procedures include sample splits, incremental samples, and rolling samples. A fairly general cointegration model is assumed, where the observed series and the cointegration error are fractionally integrated processes. Under the alternative hypothesis, the tests converge to the supremum of a chi-squared distribution. A Monte Carlo simulation is used to evaluate the finite sample performance of the tests.

Highlights:

- *Extension of a residual-based test for fractional cointegration.*
- *Incorporation of potential changes in long-run equilibrium through subsampling.*
- *Subsampling techniques include split samples, incremental, and rolling samples.*
- *The proposed subsample tests exhibit higher power than the full-sample test.*

Keywords: Fractional Cointegration, Long Memory, Monte Carlo, Persistence Breaks, Structural Breaks, Subsample Analysis

JEL classification: C12, C32

1 Introduction

Since the influential works of Engle and Granger (1987) and Johansen (1988) on cointegration, numerous papers have explored long-run equilibria between time series, including cases where the error is a fractionally integrated process. Fractional cointegration has been applied to various fields, such as exchange rates, stock volatility, and commodity prices (e.g., Cheung and Lai (1993); Robinson and Yajima (2002); Christensen and Nielsen (2006)).

The impact of structural breaks on cointegration relationships has been widely studied in the literature (e.g., Andrews et al. (1996); Inoue (1999); Johansen et al. (2000)). Moreover, studies show that long memory time series can exhibit changes in persistence over time (e.g., Beran and Terrin (1996, 1999); Kim (2000); Sibbertsen and Kruse (2009)).

For these reasons, structural breaks and shifts in persistence may disrupt long-run equilibria. In this paper, a subsample testing framework for fractional cointegration is proposed, accommodating fixed or time-varying relationships. Such subsample testing is particularly relevant when the null hypothesis of no cointegration cannot be rejected for the complete sample, as it allows practitioners to identify potential cointegration within specific periods. To this end, the residual-based test proposed by Wang et al. (2015) is extended to detect changes in fractional cointegration relationships. Following Davidson and Monticini (2010), the proposed methodology employs sample splitting as well as incremental and rolling subsampling, ensuring that subsamples grow with the complete sample to maintain the consistency of the tests. A similar analysis was recently performed by Rodrigues et al. (2024), who generalize the Hassler and Breitung (2006) test for fractional cointegration.

The remainder of the paper is organized as follows. The underlying model and tests are presented in Section 2, while the results of the Monte Carlo study supporting the superiority of the subsample tests over the full-sample test are discussed in Section 3. Concluding remarks are given in Section 4.

2 Tests for fractional cointegration in subsamples

Consider the following bivariate model, as presented in Wang et al. (2015),

$$\begin{aligned} y_t &= \beta x_t + \Delta^{-\delta}\{\upsilon_{1t}\} \\ x_t &= \Delta^{-d}\{\upsilon_{2t}\}, \end{aligned} \tag{1}$$

where $t = 1, 2, \ldots, T$, $\Delta = 1 - L$, L *denotes the lag operator, and* $\Delta^{-d} = \sum_{j=0}^{\infty} b_j(-d)L^j$, *where* $b_j(d) = \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)}$ *for* $d \neq 0, -1, -2, \ldots,$ *and* $\Gamma(\cdot)$ *is the Gamma function with* $\Gamma(0)/\Gamma(0) = 1$ *and* $\Gamma(\omega) = \infty$ *for* $\omega = 0, -1, -2, \ldots$. *In this model,* x_t *and* y_t *are non-stationary long memory processes, integrated of order* $d > 1/2$. *The parameter* $\delta \leq d$ *determines the memory of the regression residuals, which are given by* $u_t = y_t - \beta x_t$. *If* $\delta < d$, x_t *and* y_t *are fractionally cointegrated. Model (1) encompasses the traditional cointegration framework when* $d = 1$ *and* $\delta = 0$. *Moreover,* $\upsilon_t = (\upsilon_{1t}, \upsilon_{2t})'$ *are white noise processes as specified in the following assumption:*

Assumption 1. *Let* $\upsilon_t = A(L)\epsilon_t$, *where* $A(L) = \sum_{j=0}^{\infty} A_j L^j$. *Further assume that*

(i) *det*$(A(1)) \neq 0$ *which implies* $\sum_{j=1}^{\infty} j\|A_j\|^2 < \infty$, *where* $\|\cdot\|$ *denotes the Euclidean norm;*

(ii) $\epsilon_t \overset{\text{iid}}{\sim} (0, \Omega)$, *where* Ω *is positive definite, and* $E\|\epsilon_t\|^q < \infty$ *for some* $q \geq 4$, $q > 2/(2d - 1)$;

(iii) $f_{ii}(0) > 0$, $i = 1, 2$, *where* $f_{ij}(0)$ *is the* (i, j)*-th element of the spectral density of* υ_t, *denoted as* $f(\lambda)$, *at frequency zero.*

The conditions in (i) and (ii) imply that x_t *and* y_t *are type-II fractionally integrated processes as defined by Marinucci and Robinson (1999), so that the functional central limit theorem from Marinucci and Robinson (2000) can be applied. The moment condition specified in (ii) is inherently satisfied for any nonstationary process under the assumption of*

Gaussianity (Robinson and Hualde (2003)). Lastly, the conditions in (ii) and (iii) imply that $f(\lambda)$ is Lip(ξ) for some $\xi > 0$, ensuring the regularity required to establish the asymptotic properties of the test statistics defined later. Assumption 1 is convenient as it is satisfied by the conventional stationary and invertible autoregressive moving average processes.

Under the null hypothesis, there is no fractional cointegration, i.e., H_0: $\delta = d$. Typically, the alternative hypothesis assumes the presence of fractional cointegration across the entire set of observations. In this study, the alternative of fractional cointegration in subsamples is of interest, implying that the memory of the cointegration residuals varies over time, making the cointegration relationship dynamic. It is defined as H_1: $\delta_t < d$ for $t = \lfloor \lambda_1 T \rfloor + 1, \ldots, \lfloor \lambda_2 T \rfloor$ and $\delta_t = d$ elsewhere, with $0 \leq \lambda_1 < \lambda_2 \leq 1$ and $\lfloor \lambda_1 T \rfloor < \lfloor \lambda_2 T \rfloor$. Given that d, δ, and $f(0)$ are unknown in practical applications, the following assumption is required to derive the estimates $\hat{d}$, $\hat{\delta}$, and $\hat{f}(0)$.

Assumption 2. *Under the null and the alternative hypotheses,*

(i) there exists a $\kappa < \infty$ so that

$$|\hat{d}| + |\hat{\delta}_t| \leq \kappa, \tag{2}$$

and for some $\eta > 0$,

$$\hat{d} = d + O_p(T^{-\eta}) \quad \text{and} \quad \hat{\delta}_t = \delta_t + O_p(T^{-\eta}), \tag{3}$$

(ii) $\hat{f}(0) \xrightarrow{p} f(0)$, where $\xrightarrow{p}$ stands for convergence in probability.

Condition (2) in (i) is not restrictive when $\hat{d}$ and $\hat{\delta}_t$ are optimizers of their respective functions over compact sets, which is typical for implicitly defined estimators (Robinson and Hualde (2003)). To fulfill Condition 3, a wide range of parametric and semi-parametric long memory estimators is available. However, estimating δ_t requires prior estimation of u_t. For this purpose, the cointegrating vector β can be estimated using either ordinary least *squares (OLS) or narrow-band least squares. For a discussion of potential issues under H_1, see Hualde and Velasco (2008).*

For fixed truncation points, λ_1 and λ_2, the residual-based test for fractional cointegration is

$$\hat{F} = F(\hat{\delta}_t, \hat{f}^{\tau}_{22}(0), \lambda_1, \lambda_2) = \frac{(\lfloor \lambda_2 T \rfloor - \lfloor \lambda_1 T \rfloor)^{-1/2} \sum_{t=\tau} \Delta^{\hat{\delta}_t} x_t}{\left(2\pi \hat{f}^{\tau}_{22}(0)\right)^{1/2}}, \tag{4}$$

where $\hat{f}^{\tau}_{22}(0)$ is defined over the subinterval $\tau = \lfloor \lambda_1 T \rfloor + 1, \ldots, \lfloor \lambda_2 T \rfloor$, and $\sum_{t=\tau}$ denotes $\sum_{t=\lfloor \lambda_1 T \rfloor+1}^{\lfloor \lambda_2 T \rfloor}$ throughout the paper. This leads to the following theorem.

Theorem 1. *Given that Assumptions 1 and 2 are satisfied and the data is generated according to the model specified in Eq. (1), it follows that $\hat{F} \Rightarrow N(0,1)$ under H_0, and $\hat{F} = O_p(T^{d-\delta_t})$ under H_1, where $\Rightarrow$ denotes weak convergence.*

The proofs of the theorems are provided in the Appendix. Since the truncation points λ_1 and λ_2 are unknown in empirical applications, the subsample testing framework proposed by Davidson and Monticini (2010) is adopted, which involves the introduction of the following sets:

$$\Lambda_S = \{\{0, 1/2\}, \{1/2, 1\}\}, \tag{5}$$

$$\Lambda_{0f} = \{\{0, s\} : s \in [\lambda_0, 1]\}, \tag{6}$$

$$\Lambda_{0b} = \{\{s, 1\} : s \in [0, 1 - \lambda_0]\}, \tag{7}$$

$$\Lambda_{0R} = \{\{s, s + \lambda_0\} : s \in [0, 1 - \lambda_0]\}, \tag{8}$$

where $\lambda_0 \in (0,1)$ is a preselected value. The set Λ_S splits the sample into two equal halves. Incremental forward and backward samples, denoted by Λ_{0f} and Λ_{0b} respectively, are constructed with a minimum length of $\lfloor \lambda_0 T \rfloor$ and a maximum length of T. This design allows for the detection of structural breaks occurring after $\lfloor \lambda_0 T \rfloor$ and before $\lfloor (1 - \lambda_0) T \rfloor$. The rolling window samples, denoted as Λ_{0R}, consist of subsets with a fixed length of $\lfloor \lambda_0 T \rfloor$, designed to identify windows of fractional cointegration with unknown start and end points.

Additionally, the split sample and rolling window subsets are augmented by the full-sample test:

$$\Lambda_{S^*} = \Lambda_S \cup \{0, 1\}, \tag{9}$$

$$\Lambda_{0R^*} = \Lambda_{0R} \cup \{0, 1\}. \tag{10}$$

The subsample fractional cointegration tests are obtained by applying the test in Eq. (4) to the sets in Eqs. (5) - (10). The split sample tests are expressed as:

$$\mathcal{F}_S = \max_{\{\lambda_1, \lambda_2\} \in \Lambda_S} F^2(\hat{\delta}_t, \hat{f}_{22}^{\tau}(0), \lambda_1, \lambda_2), \tag{11}$$

$$\mathcal{F}_{S^*} = \max_{\{\lambda_1, \lambda_2\} \in \Lambda_{S^*}} F^2(\hat{\delta}_t, \hat{f}_{22}^{\tau}(0), \lambda_1, \lambda_2). \tag{12}$$

Incremental forward and backward tests are defined as:

$$\mathcal{F}_{I_f}(\lambda) = \max_{\lambda_0 \leq \lambda \leq 1} F^2(\hat{\delta}_t, \hat{f}_{22}^{\tau}(0), 0, \lambda), \tag{13}$$

$$\mathcal{F}_{I_b}(\lambda) = \max_{0 \leq \lambda \leq 1-\lambda_0} F^2(\hat{\delta}_t, \hat{f}_{22}^{\tau}(0), \lambda, 1), \tag{14}$$

while rolling window tests are given by:

$$\mathcal{F}_R(\lambda) = \max_{0 \leq \lambda \leq 1-\lambda_0} F^2(\hat{\delta}_t, \hat{f}_{22}^{\tau}(0), \lambda, \lambda + \lambda_0), \tag{15}$$

$$\mathcal{F}_{R^*}(\lambda) = \max_{\{\lambda_1, \lambda_2\} \in \Lambda_{0R^*}} F^2(\hat{\delta}_t, \hat{f}_{22}^{\tau}(0), \lambda_1, \lambda_2). \tag{16}$$

These test statistics can be generalized as:

$$\mathcal{F}_K(\lambda_1, \lambda_2) := \max_{\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2} F^2(\hat{\delta}_t, \hat{f}_{22}^{\tau}(0), \lambda_1, \lambda_2), \quad K = S, S^*, I_f, I_b, R, R^*.$$

Based on Theorem 1 and the findings in Davidson and Monticini (2010), the limit distribution for the test statistics in Eqs. (11) - (16) can be stated.

Theorem 2. *Under Assumptions 1 and 2, and given that the data is generated from a model as in Eq. (1), then under the null hypothesis of no fractional cointegration, it follows that*

$$\mathcal{F}_K(\lambda_1, \lambda_2) \Rightarrow \sup_{\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2} \chi_1^2(\lambda_1, \lambda_2), \quad K = S, S^*, I_f, I_b, R, R^*.$$

3 Monte Carlo simulation

In this section, the finite sample properties of the proposed tests are obtained by Monte Carlo simulations. To ensure comparability with Wang et al. (2015), the approximate maximum likelihood estimator by Beran (1995) is used to determine the memory parameter d*. This estimator is defined as the minimizer of the objective function* $S(d) = \sum_{t=\lfloor\lambda_1 T\rfloor+2}^{\lfloor\lambda_2 T\rfloor} e_t^2(d)$*, where* $e_t(d) = \sum_{j=0}^{t-1} a_j(d) x_{t-j}$*, and* $a_j(d) = \sum_{k=0}^{min(m,j)} (-1)^k \binom{m}{k} b_{j-k}(-\gamma)$*. Here,* $\gamma = d - m$*,* $\gamma \in (-1/2, 1/2)$*,* $m \geq 0$ *is an integer defining the degree of differencing required to achieve stationarity, and* b_j *is defined in Section 2. The estimator is* $T^{1/2}$*-consistent for any real* $d > -1/2$ *and is asymptotically normal. Furthermore, the consistent estimator* $\hat{f}_{22}^{\tau}(0) = \frac{1}{2\pi(\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor)} \sum_{t=\tau} \left(\Delta^{\hat{d}} x_t\right)^2$ *is applied, while the cointegrating vector is estimated using OLS from a model excluding a constant term. Finally, the memory parameter* δ_t *of the residuals is estimated with the same method as* d*.*

Model (1) is the data-generating process with $\beta = 1$*,* $\upsilon_t = (\upsilon_{1t}, \upsilon_{2t})'$ *are Gaussian white noise processes with* $E[\upsilon_t] = 0$*,* $\mathrm{Var}(\upsilon_{1t}) = \mathrm{Var}(\upsilon_{2t}) = 1$ *and* $\mathrm{Cov}(\upsilon_{1t}, \upsilon_{2t}) = \rho = 0$*. The initial values of* υ_{1t} *and* υ_{2t} *are set to zero as in Wang et al. (2015). The sample sizes are* $T = 250$ *and* 500*, with the fractional parameters* $d = 0.9$ *and* 0.7*. For each* d*, let* $\delta_t = d$*,* $d - 0.4$ *and* $d - 0.6$*. Throughout all simulations,* $\lambda_0 = 0.5$ *for both the incremental and rolling tests. The finite sample properties are obtained via* 5,000 *Monte Carlo repetitions.*

In Table 1 the critical values for the tests in Eqs. (11) - (16) are shown. Since the proposed tests depend only on the estimation error $\delta_t - \hat{\delta}_t$ *under* H_0 *and not on the specific values of* d *and* δ_t*, the critical values are averaged over* $d = \{0.6, 0.7, 0.8, 0.9, 1\}$*.*

The empirical size of the subsample tests and the full-sample test by Wang et al. (2015), denoted as $\mathcal{F}_W$*, is shown in Table 2. The full-sample test is oversized for all nominal significance levels and sample sizes, whereas the empirical size of the subsample tests is close to the nominal size, except for the incremental backward test.*

Table 1: Critical values at 1%, 5% and 10% for the subsample tests, averaged over $d = \{0.6, 0.7, 0.8, 0.9, 1\}$ with 5,000 Monte Carlo repetitions.

	$\mathcal{F}_S$	$\mathcal{F}_{S^*}$	$\mathcal{F}_{I_f}$	$\mathcal{F}_{I_b}$	$\mathcal{F}_R$	$\mathcal{F}_{R^*}$	$\mathcal{F}_S$	$\mathcal{F}_{S^*}$	$\mathcal{F}_{I_f}$	$\mathcal{F}_{I_b}$	$\mathcal{F}_R$	$\mathcal{F}_{R^*}$
	$T = 250$						$T = 500$					
1%	26.457	26.714	24.647	17.091	47.088	47.228	16.973	17.268	17.252	11.261	27.178	27.178
5%	12.322	12.490	12.223	7.847	22.220	22.220	8.860	8.979	9.339	6.055	14.746	14.759
10%	8.162	8.295	8.218	5.327	14.654	14.655	6.163	6.268	6.669	4.247	10.572	10.585

Table 2: Empirical size of full-sample and subsample tests at nominal significance levels of 1%, 5% and 10%.

		$\mathcal{F}_W$	$\mathcal{F}_S$	$\mathcal{F}_{S^*}$	$\mathcal{F}_{I_f}$	$\mathcal{F}_{I_b}$	$\mathcal{F}_R$	$\mathcal{F}_{R^*}$	$\mathcal{F}_W$	$\mathcal{F}_S$	$\mathcal{F}_{S^*}$	$\mathcal{F}_{I_f}$	$\mathcal{F}_{I_b}$	$\mathcal{F}_R$	$\mathcal{F}_{R^*}$
d		$T = 250$							$T = 500$						
	1%	0.038	0.007	0.007	0.009	0.004	0.009	0.009	0.026	0.008	0.008	0.008	0.004	0.010	0.010
0.9	5%	0.107	0.044	0.045	0.048	0.025	0.047	0.047	0.070	0.039	0.040	0.046	0.019	0.043	0.043
	10%	0.154	0.091	0.091	0.099	0.053	0.095	0.095	0.131	0.100	0.100	0.101	0.046	0.100	0.100
	1%	0.040	0.014	0.014	0.010	0.021	0.013	0.013	0.026	0.012	0.012	0.008	0.023	0.011	0.011
0.7	5%	0.097	0.055	0.055	0.046	0.082	0.056	0.056	0.077	0.057	0.056	0.044	0.082	0.053	0.053
	10%	0.153	0.102	0.101	0.093	0.135	0.100	0.100	0.134	0.120	0.116	0.100	0.157	0.108	0.108

For the empirical rejection frequencies under H_1, three scenarios are considered.

Scenario 1: *Constant fractional cointegration relationship throughout the sample.*

Scenario 2: *Constant fractional cointegration in the first half, no cointegration in the second half:*

$$\begin{cases} \delta_t < d, & for\ t = 1, ..., 0.5\ T, \\ \delta_t = d, & for\ t = 0.5\ T + 1, \ldots, T. \end{cases}$$

Scenario 3: *No cointegration in the first half, constant fractional cointegration in the second half:*

$$\begin{cases} \delta_t = d, & for\ t = 1, ..., 0.5\ T, \\ \delta_t < d, & for\ t = 0.5\ T + 1, \ldots, T. \end{cases}$$

The empirical power of the tests at a 5% *significance level is presented in Table 3. Compared to the full-sample test, the subsample tests exhibit higher power, even under the constant fractional cointegration relationship in* ***Scenario 1****. In* ***Scenarios 2*** *and* ***3****, the backward and forward incremental tests have lower power than the full-sample test. This is because the forward test is designed to detect changes from fractional cointegration to no fractional cointegration (****Scenario 2****), while the backward test is designed for* ***Scenario 3****. Lastly, the power of all tests increases with larger sample sizes and greater differences* $d - \delta_t$.

Additionally, the critical values, size, and power were computed for $\rho = 0.5$. *Since the results are qualitatively similar to those with uncorrelated innovations, they are not reported.*

Table 3: Empirical power of full-sample and subsample tests at a nominal significance level of 5%.

d	δ_t	$\mathcal{F}_W$	$\mathcal{F}_S$	$\mathcal{F}_{S^*}$	$\mathcal{F}_{I_f}$	$\mathcal{F}_{I_b}$	$\mathcal{F}_R$	$\mathcal{F}_{R^*}$	$\mathcal{F}_W$	$\mathcal{F}_S$	$\mathcal{F}_{S^*}$	$\mathcal{F}_{I_f}$	$\mathcal{F}_{I_b}$	$\mathcal{F}_R$	$\mathcal{F}_{R^*}$
Scenario 1															
d	δ_t	$T = 250$							$T = 500$						
0.9	0.5	0.798	0.828	0.829	0.841	0.834	0.820	0.820	0.847	0.915	0.915	0.934	0.931	0.931	0.932
	0.3	0.930	0.965	0.966	0.980	0.980	0.978	0.979	0.950	0.988	0.988	0.998	0.998	0.997	0.998
0.7	0.3	0.807	0.834	0.838	0.839	0.889	0.838	0.838	0.851	0.921	0.922	0.942	0.960	0.947	0.947
	0.1	0.922	0.968	0.970	0.982	0.987	0.984	0.984	0.950	0.989	0.990	0.998	0.999	0.998	0.998
Scenario 2															
d	δ_t	$T = 250$							$T = 500$						
0.9	0.5	0.358	0.581	0.590	0.672	0.055	0.554	0.556	0.378	0.705	0.716	0.781	0.070	0.696	0.697
	0.3	0.459	0.816	0.826	0.886	0.102	0.821	0.822	0.459	0.890	0.902	0.945	0.113	0.906	0.909
0.7	0.3	0.369	0.581	0.587	0.670	0.176	0.567	0.568	0.393	0.714	0.729	0.793	0.234	0.717	0.718
	0.1	0.456	0.807	0.821	0.873	0.245	0.816	0.817	0.497	0.892	0.903	0.948	0.331	0.918	0.919
Scenario 3															
d	δ_t	$T = 250$							$T = 500$						
0.9	0.5	0.317	0.669	0.666	0.151	0.723	0.673	0.673	0.323	0.788	0.786	0.194	0.819	0.800	0.800
	0.3	0.412	0.875	0.874	0.212	0.918	0.872	0.872	0.426	0.929	0.928	0.286	0.951	0.930	0.930
0.7	0.3	0.347	0.676	0.673	0.163	0.765	0.619	0.619	0.364	0.788	0.786	0.217	0.863	0.759	0.759
	0.1	0.448	0.877	0.877	0.244	0.929	0.859	0.859	0.464	0.931	0.931	0.315	0.964	0.924	0.924

4 Conclusion

In this paper, residual-based tests for fractional cointegration are proposed that can accommodate changes in the underlying long-run equilibrium. The tests allow for unknown integration orders for both the observed time series and the cointegration residuals. Nevertheless, valid plug-in estimation is obtained under mild assumptions. The proposed tests are shown to converge to the supremum of a chi-squared distribution and to have higher power than their full-sample competitor, even in the presence of a constant fractional cointegration relationship over the complete set of observations. The approach presented in this *paper offers greater flexibility compared to the conventional constant fractional cointegration assumption, making it suitable for a wide range of applications.*

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Appendix

Proof of Theorem 1. The proof follows the lines in Wang et al. (2015), while their arguments are localized to the interval $\lfloor \lambda_1 T \rfloor + 1, \ldots, \lfloor \lambda_2 T \rfloor$. It relies on the fact that when T diverges, $\lfloor \lambda_2 T \rfloor - \lfloor \lambda_1 T \rfloor$ also diverges as long as $0 \leq \lambda_1 < \lambda_2 \leq 1$.

Let $g(d, z_t) = \Delta^d z_t$, then $g(d, z_t) = \sum_{i=0}^{t-1} b_i(d) z_{t-i}$ if $z_t = 0$ for $t \leq 0$ and $g^{(r)}(d, z_t) = \sum_{i=1}^{t-1} b_i^{(r)}(d) z_{t-i}$, where $b_i^{(r)}(d) = (\partial^r / \partial d^r) b_i(d)$. Using a Taylor expansion around d, with some constant R, it follows that

$$
\begin{aligned}
&(\lfloor \lambda_2 T \rfloor - \lfloor \lambda_1 T \rfloor)^{-1/2} \sum_{t=\tau} \Delta^{\hat{\delta}_t} x_t - (\lfloor \lambda_2 T \rfloor - \lfloor \lambda_1 T \rfloor)^{-1/2} \sum_{t=\tau} \Delta^{\delta_t} x_t \\
&= (\lfloor \lambda_2 T \rfloor - \lfloor \lambda_1 T \rfloor)^{-1/2} \sum_{t=\tau} \Big(g(d - \hat{\delta}_t; \upsilon_{2t}) - g(d - \delta_t; \upsilon_{2t}) \Big) \\
&= \frac{1}{\sqrt{(\lfloor \lambda_2 T \rfloor - \lfloor \lambda_1 T \rfloor)}} \sum_{r=1}^{R-1} \frac{(\delta_t - \hat{\delta}_t)^r}{r!} \sum_{t=\tau} g^{(r)}(d - \delta_t; \upsilon_{2t}) \\
&+ \frac{(\delta_t - \hat{\delta}_t)^R}{R! \sqrt{(\lfloor \lambda_2 T \rfloor - \lfloor \lambda_1 T \rfloor)}} \sum_{t=\tau} g^{(R)}(d - \tilde{\delta}_t; \upsilon_{2t}) \\
&= \begin{cases} o_p(1), & \text{under } H_0, \\ o_p(T^{d-\delta_t}), & \text{under } H_1, \end{cases}
\end{aligned}
\tag{A.1}
$$

where $\tilde{\delta}_t \in (\min(\delta_t, \hat{\delta}_t), \max(\delta_t, \hat{\delta}_t))$.

It can be derived that $\left|b_i^{(r)}(0)\right| \leq \frac{K_r(\log(i+1))^{r-1}}{i-r+1}$, for $i \geq r$ and some constant K_r.

Under H_0, Var $\left(\sum_{t=\tau} g^{(r)}(0, \upsilon_{2t})\right) = O(T(\log T)^{2r})$. This result follows from a modification of the proof of (C.8) in Robinson and Hualde (2003), using Assumption 1 and the bound of $\left|b_i^{(r)}(0)\right|$. Consequently, $\sum_{t=\tau} g^{(r)}(0, \upsilon_{2t}) = O_p(T^{1/2}(\log T)^r)$. The first term of (A.1) is $O_p(T^{-\eta}\log T)$, which follows directly from Assumption 2. In addition, by the bound of $\left|b_i^{(r)}(0)\right|$, $g^{(R)}(\delta - \tilde{\delta}; \upsilon_{2t}) = O_p(T^{1/2})$ uniformly in T, so that the second term in (A.1) is $O_p(T^{1-R\eta})$. As long as $R > (1+\eta)/\eta$, then (A.1) is $O_p(T^{-\eta}\log T) = o_p(1)$. Since $\hat{f}_{22}^{\tau}(0)$ consistently estimates $f_{22}^{\tau}(0)$ according to Assumption 2, it follows that $F(\delta, f_{22}^{\tau}(0), \lambda_1, \lambda_2) - \hat{F} = o_p(1)$. When $\delta = d$, Theorem 1 in Marinucci and Robinson (2000) implies that $F(\delta, f_{22}^{\tau}(0), \lambda_1, \lambda_2) = \frac{(\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor)^{-1/2}\sum_{t=\tau}\upsilon_{2t}}{\left(2\pi f_{22}^{\tau}(0)\right)^{1/2}} \Rightarrow N(0,1)$, and therefore, $\hat{F} \Rightarrow N(0,1)$.

Under H_1, Var $\left(\sum_{t=\tau} g^{(r)}(d-\delta_t, \upsilon_{2t})\right) = O(T^{2(d-\delta_t)+1}(\log T)^{2r})$, and $\sum_{t=\tau} g^{(r)}(d-\delta_t, \upsilon_{2t}) = O_p(T^{d-\delta_t+1/2}(\log T)^r)$, following the proof of (C.8) in Robinson and Hualde (2003). The first term of (A.1) is $O_p(T^{d-\delta_t-\eta}\log T)$ and by Lemma C.4 of Robinson and Hualde (2003), $\text{Var}(g^{(R)}(d-\delta_t, \upsilon_{2t})) \leq K\sum_{i=1}^{t-1}(\log i)^{2R} i^{2(d-\delta_t)-2}$ for some constant K. Consequently, $\sum_{t=\tau} g^{(R)}(d-\delta_t, \upsilon_{2t}) = O_p((\log T)^R T^{d-\delta_t})$ and for the second term in (A.1) it holds $O_p((\log T)^R T^{-\eta R+d-\delta_t})$. If $R > \frac{d-\delta_t}{\eta}$, then this becomes $O_p(T^{d-\delta_t-\eta}\log T)$. As a result, (A.1) is $o_p(T^{d-\delta_t})$. It follows that $F(\delta_t, f_{22}^{\tau}(0), \lambda_1, \lambda_2) - \hat{F} = o_p(T^{d-\delta_t})$. Under Assumption 1 and Theorem 1 of Marinucci and Robinson (2000), $F(\delta_t, f_{22}^{\tau}(0), \lambda_1, \lambda_2) = O_p(T^{d-\delta_t})$ and finally, $\hat{F} = O_p(T^{d-\delta_t})$, which completes the proof. □

Proof of Theorem 2. The proof follows directly from Theorem 1 and the arguments in Davidson and Monticini (2010). □

Data availability

No data was used for the research described in the article.

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